

Applications on differential subordination involving linear operator

Abdul Rahman Salman Juma

Department of Mathematics, University of Anbar, Ramadi, Iraq

E-mail: absa662004@yahoo.com

Abstract

In this paper we introduce and investigate some subclasses of strongly close-to-convex functions associated with the linear operator of meromorphic p -valently functions and study several inclusion relationships with some properties of this operator.

Key Words: linear operator, meromorphic functions, differential subordination, strongly Close-to-convex functions, p -valently functions.

AMS Subject Classification: Secondary 30C45.

1.Introduction

Let Σ_p denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_{k+p} z^{k+p}, \quad (1-1)$$

which are analytic and p -valently in the punctured unit disk $\mathcal{U}^* = \{z: z \in \mathbb{C}, 0 < |z| < 1\}$

$$= \mathcal{U} - 0.$$

If $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f < g$ or $f(z) < g(z)$, if there exists a Schwarz function $w(z)$ which (by definition) is analytic

in $\mathcal{U}$ such that $f(z) = g(w(z))$.

A function $f(z) \in \Sigma_p$ is said to be p -valent meromorphic starlike of order α ($0 \leq \alpha < p$)

if it satisfies

$$\operatorname{Re} \left\{ -\frac{zf'(z)}{f(z)} \right\} \geq \alpha \quad (z \in \mathcal{U}) \quad (1-2)$$

and the class of such functions is defined by $MS^*(\alpha)$.

Also a function $f(z) \in \Sigma_p$ is said to be p -valent meromorphic convex of order

α ($0 \leq \alpha < p$) if it satisfies

$$\operatorname{Re} \left\{ -\left(1 + \frac{zf''(z)}{f'(z)}\right) \right\} \geq \alpha \quad (z \in \mathcal{U}) \quad (1-3)$$

and the class of such functions is defined by $MK(\alpha)$.

Let $f(z) \in \Sigma_p$ and $g(z) \in MS^*(\alpha)$. Then $f(z) \in MC(\alpha, \beta)$ if and only if

$$\operatorname{Re} \left\{ -\frac{zf'(z)}{g(z)} \right\} > \beta \quad (z \in \mathcal{U}), \quad (1-4)$$

where $0 \leq \alpha < p$ and $0 \leq \beta < p$. Such functions are called close-to-convex functions of

order β and type α in $\mathcal{U}_{(1,2)}$.

Further, a function $f(z) \in \Sigma_p$ is called p -valently meromorphic strongly starlike of order γ ($0 \leq \gamma < p$) and type α ($0 \leq \alpha < p$) in $\mathcal{U}$ if it satisfies

$$\left| \arg \left(-\frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi}{2} \gamma \quad (z \in \mathcal{U}). \quad (1-5)$$

If $f(z) \in \Sigma_p$ satisfies

$$\left| \arg \left(-\left(1 + \frac{zf''(z)}{f'(z)} \right) - \alpha \right) \right| < \frac{\pi}{2} \gamma \quad (z \in \mathcal{U}),$$

for some γ ($0 \leq \gamma < p$) and α ($0 \leq \alpha < p$), then f is called p -valently meromorphic

strongly convex of order γ and type α in $\mathcal{U}$ and denoted by $MC(\gamma, \alpha)$. We note that the

classes mentioned above are the familiar classes which have been studied by many authors^(3, 4)

For a function $f(z) \in \Sigma_p$ given by (1-1), we define a linear operator D^n by

$$D^0 f(z) = f(z)$$

$$\begin{aligned} D^1 f(z) &= z^{-p} (z^{p+1} f(z))' \\ &= z^{-p} \\ &\quad + \sum_{k=0}^{\infty} (2p + k + 1) a_{k+p} z^{k+p} \end{aligned}$$

and

$$\begin{aligned} D^n f(z) &= D(D^{n-1} f(z)) \\ &= z^{-p} (z^{p+1} D^{n-1} f(z))' \\ &= z^{-p} \\ &\quad + \sum_{k=0}^{\infty} (2p + k + 1)^n a_{k+p} z^{k+p}. \quad (N \in \mathbb{N}) \end{aligned} \quad (1-6)$$

Using the relation (1-6), it is easy to verify that

$$(D^n f(z))' = D^{n+1} f(z) - (p+1) D^n f(z). \quad (1-7)$$

Also, we note that $MS^*(\gamma, \alpha)$ is another form of function studied by Liu and Srivastava⁽⁵⁾, Srivastava and Patel⁽⁶⁾ who introduce several inclusion relationships by using various subclasses of meromorphic p -valent functions. A special cases of linear operator D^n for $p=1$ studied by Uralegaddi and Somanatha⁽⁷⁾, Aouf and Hossen⁽⁸⁾, and got interesting results by using the operator D^n .

For $n \in \mathbb{N}$, let $MC_p^{n+1}(\alpha, \beta, \gamma, A, B)$ be the class of functions $f(z) \in \Sigma_p$ satisfying the condition:

$$\begin{aligned} \left| -\arg \left(\frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)} - \gamma \right) \right| &< \frac{\pi}{2} \delta \quad (0 \leq \gamma < p, 0 \leq \delta < p; z \in \mathcal{U}), \end{aligned} \quad (1-8)$$

for some $g(z) \in S_p^{n+1}(\alpha, A, B)$, where

$$\begin{aligned} S_p^{n+1}(\alpha, A, B) &= \left\{ g: \frac{1}{p+\alpha} \left(\frac{z(D^{n+1} g(z))'}{D^{n+1} g(z)} - \alpha \right) < \frac{1+Az}{1+Bz} \right\} \end{aligned} \quad (1-9)$$

($0 \leq \alpha < p, -1 \leq B \leq A \leq 1, z \in \mathcal{U}$ and $g \in \Sigma_p$) and the functions f belonging to this class is called strongly close-to-convex function.

In this study and by using the technique of Cho, we find some argument properties of functions belonging to Σ_p which include inclusion relationship and we obtain some interesting results for the functions class $MC_p^{n+1}(\alpha, \beta, \gamma, A, B)$ which we have defined here

by the operator D^n .

In order to obtain our results, we need each of the following lemmas.

Lemma 1.1⁽⁹⁾: Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $\operatorname{Re} \{ \varepsilon h(z) + \eta \} >$

0 ($\varepsilon, \eta \in \mathbb{C}$). If $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\varepsilon p(z) + \eta} < h(z) \quad (z \in \mathcal{U}),$$

implies $p(z) < h(z)$ ($z \in \mathcal{U}$).

Lemma 1.2⁽¹⁰⁾: Let $h(z)$ be convex univalent in $\mathcal{U}$ and $w(z)$ be analytic in $\mathcal{U}$ with $\operatorname{Re} \{ w(z) \} \geq 0$. If $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = h(0)$, then

$$p(z) + w(z)zp'(z) < h(z) \quad (z \in \mathcal{U}),$$

implies $p(z) < h(z)$ ($z \in \mathcal{U}$).

Lemma 1.3⁽⁴⁾: Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. If there exist two points z_1, z_2 in $\mathcal{U}$ such that

$$-\frac{\pi}{2}\alpha_1 = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2}\alpha_2 \quad (1-10)$$

for some α_1, α_2 ($\alpha_1, \alpha_2 > 0$) and for all z ($|z| < |z_1| = |z_2|$), then we have

$$\frac{zp'(z_1)}{p(z_1)} = -i \frac{\alpha_1 + \alpha_2}{2} m$$

and

$$\frac{zp'(z_2)}{p(z_2)} = -i \frac{\alpha_1 + \alpha_2}{2} m, \quad (1-11)$$

Where $m \geq \frac{1-|c|}{1+|c|}$ and

$$= i \tan \frac{\pi}{4} \left(\frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2} \right). \quad (1-12)$$

2. Main Results

We first derive the following with use of Lemma 1.1.

Proposition 2.1: Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $\operatorname{Re} \{ h(z) \} > 0$.

If a function $f(z) \in \Sigma_p$ satisfies the following condition:

$$\frac{1}{p + \alpha} \left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right) < h(z),$$

then

$$\frac{1}{p + \alpha} \left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha \right) < h(z).$$

$$(0 \leq \alpha < p; z \in \mathcal{U})$$

Proof. Let

$$p(z) = \frac{1}{p + \alpha} \left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha \right). \quad (2-1)$$

Then $p(z)$ is analytic function in $\mathcal{U}$ with $p(0) = 1$. By using (1-7), we obtain

$$p + 1 + \alpha + \frac{(p + \alpha)p(z)}{D^{n+1}f(z)} = \frac{D^n f(z)}{D^n f(z)}. \quad (2-2)$$

Differentiating Logarithmically with respect to z and multiplying by , we get

$$p(z) + \frac{zp'(z)}{p+1+\alpha+(p+\alpha)p(z)} = \frac{1}{p+\alpha} \left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right).$$

Now, by using Lemma 1.1, we obtain that

$$\frac{1}{p+\alpha} \left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha \right) < h(z),$$

deduce that $p(z) < h(z)$.

Setting $h(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B \leq A \leq 1$), in Lemma 2.1, we obtain

Corollary 2.1: For $n \in \mathbb{N}$ and $p \in \{1, 2, \dots\}$ we have

$$S_p^{n+1}(\alpha, A, B) \subset S_p^n(\alpha, A, B).$$

Proposition 2.2: Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $\operatorname{Re}\{h(z)\} > 0$. If

$$\frac{1}{p+\alpha} \left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right) < h(z),$$

then

$$\frac{1}{p+\alpha} \left(\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta f(z)} - \alpha \right) < h(z),$$

$$(0 \leq \alpha < p; z \in \mathcal{U})$$

where

$$\mathbb{I}_\theta f(z) = \frac{\theta-p}{z^\theta} \int_0^z t^{\theta-1} f(t) dt \quad (\theta \geq 0) \quad (2-3)$$

$$z(D^{n+1}\mathbb{I}_\theta f(z))' = (\theta-p)((D^{n+1}f(z))' - \theta((D^{n+1}f(z)))'. \quad (2-4)$$

$$p(z) = \frac{1}{p+\alpha} \left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right),$$

$p(z)$ is analytic function in $\mathcal{U}$ with $p(0) = 1$. Then, by (2-4), we get

$$\theta + \alpha + (p+\alpha)p(z) = (\theta-p) \frac{D^{n+1}f(z)}{D^{n+1}\mathbb{I}_\theta f(z)}. \quad (2-5)$$

By differentiating (2-5) logarithmically with respect to z and multiplying by z , we have

$$p(z) + \frac{zp'(z)}{\theta + \alpha + (p+\alpha)p(z)} = \frac{1}{p+\alpha} \left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right).$$

Thus, by Lemma 1.1, we get

$$\frac{1}{p+\alpha} \left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right) < h(z).$$

Taking $h(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B \leq A \leq 1$), in Proposition 2.2, we obtain

Corollary 2.2:

If $f(z) \in S_p^{n+1}(\alpha, A, B)$, then $\mathbb{I}_\theta(f) \in S_p^{n+1}(\alpha, A, B)$

Hence on Applying Proposition 2.2 we prove the following theorem

Theorem 2.1: Let $f(z) \in \Sigma_p$ and $0 \leq \delta_1, \delta_2 \leq p, 0 \leq \alpha < p$. If

$$-\frac{\pi}{2}\delta_1 < \arg\left(-\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma\right) < \frac{\pi}{2}\delta_2$$

for some $g(z) \in S_p^{n+1}(\alpha, A, B)$, then

Proof. From (2-3), we have

$$-\frac{\pi}{2}\beta_1 < \arg\left(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma\right) < \frac{\pi}{2}\beta_2,$$

where β_1 and β_2 ($0 < \beta_1, \beta_2 \leq p$) are the solution of the equations:

$$\delta_1 = \begin{cases} \beta_1 + \frac{\pi}{2} \tan^{-1} \left\{ \frac{(\beta_1 + \beta_2)(1 - |c|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p+\alpha)(1+A)}{(1+B)} + p + 1 + \alpha \right) (1 + |c|) + (\beta_1 + \beta_2) \sin \frac{\pi}{2} t_1} \right\} & B \neq -1, \\ \beta_1 & B = -1, \end{cases}$$

(2-6)

$$\delta_2 = \begin{cases} \beta_2 + \frac{\pi}{2} \tan^{-1} \left\{ \frac{(\beta_1 + \beta_2)(1 - |c|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p+\alpha)(1+A)}{(1+B)} + p + 1 + \alpha \right) (1 + |c|) + (\beta_1 + \beta_2) \sin \frac{\pi}{2} t_1} \right\} & B \neq -1, \\ \beta_2 & B = -1, \end{cases}$$

(2-7)

where c is given by (1-12) and $t_1 = \frac{\pi}{2} \sin^{-1} \left(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB) + (p+1+\alpha)(1-B^2)} \right)$.

Proof. Let

$$p(z) = \frac{1}{p + \alpha} \left(\frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)} - \gamma \right). \quad (2-8)$$

By using (1-7) and simple calculations, we have

$$\begin{aligned} & [(p + \gamma)(p(z) - \gamma)] D^n g(z) \\ &= D^{n+1} f(z) \\ &- (p + 1) D^n f(z). \end{aligned} \quad (2-9)$$

Take the derivative of both sides (2-9) and multiply by z , we get

$$\begin{aligned} & (p + \gamma) z p'(z) D^n g(z) + [(p + \gamma)p(z) - \gamma] z (D^n g(z))' = z (D^{n+1} f(z))' - \\ & (p + 1) (D^n f(z))' \end{aligned} \quad (2-10)$$

Since $g(z) \in S_p^{n+1}(\alpha, A, B)$, then we have $g(z) \in S_p^n(\alpha, A, B)$. (by Corollary 2.1)

Now, put

$$\begin{aligned} q(z) &= p \frac{1}{p + \alpha} \left(\frac{z(D^n g(z))'}{D^n g(z)} - \alpha \right). \text{ By using (1-7), we get} \\ (p + \alpha)q(z) + \alpha + p + 1 &= \frac{D^{n+1} g(z)}{D^n g(z)}. \end{aligned} \quad (2-11)$$

Therefore, by (2-10), (2-11), we have

$$\begin{aligned} & \frac{1}{p + \alpha} \left(\frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} - \gamma \right) \\ &= p(z) + \frac{z p'(z)}{(p + \alpha)q(z) + \alpha + p + 1}. \end{aligned}$$

Making use the result of Silverman and Silvia ⁽¹¹⁾, we obtain

$$\left| q(z) - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \quad (z \in \mathcal{U}; B \neq -1) \quad (2-12)$$

and

$$\operatorname{Re} \{q(z)\} > \frac{1 - A}{2} \quad (z \in \mathcal{U}; B \neq -1). \quad (2-13)$$

By (2-12), (2-13), we get

$$(p + \alpha)q(z) + p + \alpha + 1 = r e^{i \frac{\pi \theta}{2}},$$

where, if $B \neq -1$, we have

$$\frac{(p+\alpha)(1-A)}{1-B}P+\alpha+1 < r$$

$$< \frac{(p+\alpha)(1+A)}{1+B}P+\alpha+1$$

$-t_1 < \varphi < t_1$, t_1 is defined in (2-8)

and if $B = -1$, we have

$$\frac{(p+\alpha)(1-A)}{2}P+\alpha+1 < r < \infty, -1 < \varphi$$

$$< 1.$$

Now by Lemma 1.2 and assumption with $w = \frac{1}{(p+\alpha)q(z)+p+\alpha+1}$, we note that $p(z)$ is analytic

.with $p(0) = 1$ and $Re \{p(z)\} > 0$ in $\mathcal{U}$

Hence by Lemma 1.3 for $z_1, z_2 \in \mathcal{U}$, such that the condition (1-10) is satisfied, then we

have (1-11) under the restriction (1-12). Now

if $B \neq -1$, we get

$$\arg \left(- \left(p(z_1) + \frac{zp'(z_1)}{(p+\alpha)q(z_1)+\alpha+p+1} \right) \right)$$

$$= -\frac{\pi}{2}\beta_1 + \arg \left(1 - i \frac{\beta_1 + \beta_2}{2} m \left(re^{i\frac{\pi}{2}} \right)^{-1} \right)$$

$$\leq -\frac{\pi}{2}\beta_2$$

$$- \tan^{-1} \left(\frac{(\beta_1 + \beta_2)m \sin \frac{\pi}{2} (1 - \varphi)}{2r + (\beta_1 + \beta_2)m \cos \frac{\pi}{2} (1 - \varphi)} \right)$$

$$\leq -\frac{\pi}{2}\beta_2$$

$$- \tan^{-1} \left(\frac{(\beta_1 + \beta_2)(1 - |c|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p+\alpha)(1+A)}{(1+B)} + p + 1 + \alpha \right) (1 + |c|) + (\beta_1 + \beta_2) \sin \frac{\pi}{2} t_1} \right)$$

$$= -\frac{\pi}{2}\delta_1,$$

and

$$\arg \left(- \left(p(z_2) + \frac{zp'(z_2)}{(p+\alpha)q(z_2)+\alpha+p+1} \right) \right) \geq$$

$$\frac{-\pi}{2}\beta_2$$

$$- \tan^{-1} \left(\frac{(\beta_1 + \beta_2)(1 - |c|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p+\alpha)(1+A)}{(1+B)} + p + 1 + \alpha \right) (1 + |c|) + (\beta_1 + \beta_2) \sin \frac{\pi}{2} t_1} \right)$$

$$= \frac{-\pi}{2}\delta_2.$$

If $B = -1$, we get

$$\arg \left(- \left(p(z_1) + \frac{zp'(z_1)}{(p+\alpha)q(z_1)+\alpha+p+1} \right) \right) \leq \frac{-\pi}{2}\beta_1$$

and

$$\arg \left(- \left(p(z_2) + \frac{zp'(z_2)}{(p+\alpha)q(z_2)+\alpha+p+1} \right) \right)$$

$$\geq \frac{\pi}{2}\beta_2.$$

There are contradictions to the assumption. Hence the proof is complete

Corollary 2.3:

$$MC_p^{n+1}(\alpha, \beta, \gamma, A, B) \subset MC_p^n(\alpha, \beta, \gamma, A, B)$$

Setting $n = 0$, $\delta_1 = \delta_2 = \delta$ in Theorem 2.1, we get:

Corollary 2.4: Let $f(z) \in \Sigma_p$. If

$$\left| \frac{z(z^{-p}(z^{p+1}f(z))')'}{(z^{-p}(z^{p+1}g(z))')'} - \gamma \right| \leq \frac{\pi}{2}\delta$$

$$\left| \arg \left(- \frac{zf(z)'}{g(z)} - \gamma \right) \right| < \frac{\pi}{2}\beta,$$

where β ($0 < \beta \leq p$) is the solution of equation:

$$\delta$$

$$= \left\{ \beta + \frac{\pi}{2} \tan^{-1} \left\{ \frac{\beta \cos \frac{\pi}{2} t_1}{\frac{(p+\alpha)(1+A)}{1+B} + p + 1 + \alpha + \beta \sin \frac{\pi}{2} t_1} \right\} \right\}$$

and $t_1 = \frac{\pi}{2} \sin^{-1} \left(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB) + (p+1+\alpha)(1-B^2)} \right).$

Theorem 2.2: Let $f(z) \in \Sigma_p$ and $0 < \delta_1, \delta_2 \leq 1, 0 \leq \gamma < 1$. If

$$-\frac{\pi}{2} \delta_1 < \arg \left(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma \right) < \frac{\pi}{2} \delta_2$$

for some $g(z) \in S_p^{n+1}(\alpha, A, B)$, then

$$-\frac{\pi}{2} \beta_1 < \arg \left(-\frac{z(D^{n+1} \mathbb{I}_\theta f(z))'}{D^{n+1} \mathbb{I}_\theta g(z)} - \gamma \right) < \frac{\pi}{2} \beta_2,$$

where $\mathbb{I}_\theta$ is defined by (2-3), and β_1, β_2 are the solutions of

$$\delta_1 = \begin{cases} \beta_1 + \frac{\pi}{2} \tan^{-1} \left\{ \frac{(\beta_1 + \beta_2)(1-|c|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p+\alpha)(1+A)}{(1+B)} + p+1+\alpha \right) (1+|c|) + (\beta_1 + \beta_2) \sin \frac{\pi}{2} t_1} \right\} & B \neq -1, \\ \beta_1 & B = -1, \end{cases} \quad (2-14)$$

and

$$\delta_2 = \begin{cases} \beta_2 + \frac{\pi}{2} \tan^{-1} \left\{ \frac{(\beta_1 + \beta_2)(1-|c|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p+\alpha)(1+A)}{(1+B)} + p+1+\alpha \right) (1+|c|) + (\beta_1 + \beta_2) \sin \frac{\pi}{2} t_1} \right\} & B \neq -1, \\ \beta_2 & B = -1, \end{cases} \quad (2-15)$$

here c is given by (1-12) and $t_2 = \frac{\pi}{2} \sin^{-1} \left(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB) + (\theta+\alpha)(1-B^2)} \right).$

Proof. Let

$$p(z) = \frac{1}{p+\alpha} \left(\frac{z(D^{n+1} \mathbb{I}_\theta f(z))'}{D^{n+1} \mathbb{I}_\theta g(z)} - \gamma \right).$$

Since $g(z) \in S_p^{n+1}(\alpha, A, B)$, then we have by Corollary 2.2 that $\mathbb{I}_\theta g(z) \in S_p^{n+1}(\alpha, A, B)$.

By (2-5), we get

$$\begin{aligned} [(p+\gamma)p(z) + \gamma] D^{n+1} \mathbb{I}_\theta g(z) &= (\theta-p)(D^{n+1} f(z))' \\ &\quad - \theta(D^{n+1} \mathbb{I}_\theta f(z)) \end{aligned}$$

and simplifying, we obtain

$$\begin{aligned} (p+\gamma)zp'(z) + [(p+\gamma)p(z) + \gamma][(p+\alpha)q(z) + \theta + \alpha] &= (\theta-p) \frac{z(D^{n+1} \mathbb{I}_\theta f(z))'}{D^{n+1} \mathbb{I}_\theta g(z)}, \end{aligned}$$

$$q(z) = \frac{1}{p+\alpha} \left(\frac{z(D^{n+1} \mathbb{I}_\theta f(z))'}{D^{n+1} \mathbb{I}_\theta g(z)} - \alpha \right).$$

Therefore,

$$\begin{aligned} \frac{1}{p+\alpha} \left(\frac{z(D^{n+1} \mathbb{I}_\theta f(z))'}{D^{n+1} \mathbb{I}_\theta g(z)} - \alpha \right) &= p(z) \\ &\quad + \frac{zp'(z)}{(p+\alpha)q(z) + \alpha + \theta}. \end{aligned}$$

Thus, by the similar way of proof Theorem 2.1, we obtain the required result and the

proof is complete.

Setting $\delta_1 = \delta_2 = \delta$ in Theorem 2.2, we obtain $B = -1$.

Corollary 2.5: Let $f(z) \in \Sigma_p$ and $0 \leq \gamma < p, 0 < \delta \leq 1$. If

$$\left| \arg \left(-\frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta$$

$$\left| \arg \left(-\frac{z(D^{n+1} \mathbb{I}_\theta f(z))'}{D^{n+1} \mathbb{I}_\theta g(z)} - \gamma \right) \right| < \frac{\pi}{2} \beta,$$

Where $\mathbb{I}_\theta$ is given by (2-5), and $\beta (0 < \beta \leq 1)$ is the solution of the equation

$$\delta = \begin{cases} \beta + \frac{\pi}{2} \tan^{-1} \left\{ \frac{\beta \cos \frac{\pi}{2} t_1}{\frac{(p+\alpha)(1+A)}{1+B} + p+1+\alpha+\beta \sin \frac{\pi}{2} t_1} \right\} & B \neq -1, \\ \beta & B = -1. \end{cases} \quad \text{REFERENCES}$$

Corollary 2. 6: If $f(z) \in MC_p^{n+1}(\gamma, \delta, \alpha, A, B)$, then $\mathbb{I}_\theta f(z) \in MC_p^{n+1}(\gamma, \delta, \alpha, A, B)$.

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