

Some Generalizations of prime modules

H. K. Al-Rikaby* F. K. Mamon**

*Department of Mathematics, College of computer sciences and mathematics,
Tikrit University

** Department of mathematics, College of Education, Tikrit university, E-mail
mano_@yahoo.com

Abstract

An R -module M is called prime if and only if $\text{ann}_R M = \text{ann}_N N$ for every non-zero submodule N of M . In this paper we introduce, essentially quasi-prime modules, essentially pseudo-prime modules and essentially semi-prime modules as a generalizations of a prime modules, and study the relationships among them. Many characterizations, properties and examples are given. Furthermore, the relation of these concepts with other class of modules are studied.

1. Introduction

Throughout this paper, we assume that every ring is commutative with identity, and all modules unital. Let M be an R -module, a non-zero submodule of M which has non-zero intersection with all non-zero submodules of M is called essential. Essential submodules play important roles in this research. An R -module M is called prime if and only if (0) is a prime submodule of $M^{(1)}$ where we called a submodule N of M a prime if $rx \in N$ for $r \in R$ and $x \in M$ then either $x \in N$ or

$rM \subseteq N^{(2)}$. A proper submodule N of M is called a primary if $rm \in N$ for $r \in R$ and $m \in M$ then either $m \in N$ or $r^n M \subseteq N$ for some positive integer $n^{(3)}$. It is clear that every prime submodule is primary. A proper submodule N of a module M is called semi-prime if $r^2 m \in N, r \in R, m \in M$, then $rm \in N^{(4)}$. Many authors generalized prime modules see^(1, 5). In this paper various generalizations of prime modules have been studied, for example essentially quasi-prime modules, essentially pseudo-prime modules and essentially

semi-prime modules. We show that every essentially quasi-prime module is essentially pseudo-prime module, but the converse is not true, so we give counter example for the converse. Also, every essentially pseudo-prime modules is essentially semi-prime module, but the converse is not true so we give counter example for the converse. Furthermore we show by examples that essentially pseudo-prime module and essentially semi-prime module are independent.

2: Essentially Quasi-prime Modules

In this section, we introduce the definition of essentially quasi-prime module and give some basic properties, examples and characterizations of this concept.

Definition 2.1 An R -module M is called an essentially quasi-prime if $\text{ann}_R N$ is a prime ideal of R for each non-zero essential submodule N of M .

Every prime R -module is an essentially quasi-prime. But the converse is not true. Since The Z -module $Z \oplus Z_3$ is an essentially quasi-prime, but not prime.

The Z -module $Z \oplus Z$ is an essentially quasi-prime. The Z -

module Z_{p^∞} is not an essentially quasi-prime, but the Z -module Z is an essentially quasi-prime. The homomorphism image of an essentially quasi-prime need not to be an essentially quasi-prime module, since $Z/6Z \cong Z_6$ is not an essentially quasi-prime Z -module. It is essay matter to see that essential submodules are inheriting the essentially quasi-prime properties.

The following proposition gives a characterization of essentially quasi-prime modules.

Proposition 2.2 Let M be an R -module. Then M is an essentially quasi-prime module if and only if $\text{ann}_R N = \text{ann}_N rN$ for each non-zero essential submodule N of M with $rN \neq (0)$, $r \in R$.

Proof: Assume that M is an essentially quasi-prime R -module, and N is a non-zero essential submodule of M , $r \in R$ with $rN \neq (0)$, then $\text{ann}_R N \subseteq \text{ann}_N rN$. Now, let $a \in \text{ann}_N rN$ then $arN = (0)$, so $ar \in \text{ann}_R N$, hence either $a \in \text{ann}_R N$ or $r \in \text{ann}_R N$. If $r \in \text{ann}_R N$, then $rN = (0)$ this is a contradiction. Thus $a \in \text{ann}_R N$. So we get $\text{ann}_R N = \text{ann}_N rN$.

Conversely; Assume that $\text{ann}_R N = \text{ann}_N rN$ for each

non-zero essential submodule N of M , with $rN \neq (0), r \in R$. Let $a, b \in R$ such that $a.b \in \text{ann}_R N$ and suppose that $b \notin \text{ann}_R N$, then $abN = (0)$ and $bN \neq (0)$. Thus $a \in \text{ann}_R bN = \text{ann}_R N$. Hence $\text{ann}_R N$ is prime ideal of R . Therefore M is essentially quasi-prime R -module.

The following corollary is a direct consequence of proposition 2.2.

Corollary 2.3. Let M be an R -module. Then M is an essentially quasi-prime R -module if and only if $\text{ann}_R(m) = \text{ann}_R(rm)$ for each a non-zero essential submodule (m) generated by $0 \neq m \in M$ with $r(m) \neq (0), r \in R$.

The following proposition gives another characterization of essentially quasi-prime module.

Proposition 2.4. Let M be an R -module. Then M is essentially quasi-prime if and only if $\text{ann}_R(m)$ is prime ideal for each non-zero essential submodule (m) generated by $0 \neq m \in M$.

Proof. Suppose that $\text{ann}_R(m)$ is prime ideal of R for each non-zero essential submodule generated by $0 \neq m \in M$. Let N be a non-zero essential submodule of M . $\text{ann}_R(N) =$

$[0: (n)]$ is a prime ideal of R for each non-zero essential submodule $(n) \subseteq N$. Hence by [Abdul-razak H.M., lemma 1.2.5]⁽⁶⁾ we get $[0: (n)]$ is prime ideal of R . therefore M is essentially quasi-prime R -module.

The converse is direct.

Proposition 2.5. Let M_1 and M_2 be two essentially quasi-prime modules such that for each non-zero essential submodule N and K of M_1 and M_2 respectively, either $\text{ann}_R(N) \subseteq \text{ann}_R(K)$ or $\text{ann}_R(K) \subseteq \text{ann}_R(N)$. Then $M = M_1 \oplus M_2$ is essentially quasi-prime R -module.

Proof. Since N is an essential submodule of M_1 and K is essential submodule of M_2 , then by [coro.5.1.7]⁽⁷⁾, $N \oplus K$ is non-zero essential submodule of M . To prove that $\text{ann}_R(N \oplus K)$ is prime ideal of R . Since $\text{ann}_R(N \oplus K) = \text{ann}_R N \cap \text{ann}_R K$, then by hypothesis, either $\text{ann}_R(N \oplus K) = \text{ann}_R N$ or $\text{ann}_R(N \oplus K) = \text{ann}_R K$. Hence $\text{ann}_R(N \oplus K)$ is prime ideal of R . Hence M is essentially quasi-prime R -module.

Recall that an R -module M is multiplication if every submodule N of M , there exists an ideal I of R such that $N = IM$ ⁽⁸⁾

The following proposition shows that in the class of multiplication modules the two concepts prime and essentially quasi-prime R -modules are equivalent.

Proposition 2.6. Let M be a multiplication R -module. Then M is essentially quasi-prime if and only if M is prime.

Proof. Suppose that M is an essentially quasi-prime R -module, and let $r \in R, m \in M$ such that $r(m) = (0)$. Since M is multiplication, then $(m) = IM$ for some ideal I of R . Hence $rIM = (0)$. That is $rI \subseteq \text{ann}_R M$. But M is essentially quasi-prime, then $\text{ann}_R(M)$ is prime ideal of R . Hence either $r \in \text{ann}_R(M)$ or $I \subseteq \text{ann}_R(M)$. Thus either $r \in \text{ann}_R(M)$ or $IM = (0)$. Hence either $r \in \text{ann}_R(M)$ or $(m) = (0)$. Thus (0) is a prime submodule. Then by ⁽¹⁾ M is prime.

The converse is direct.

Since every cyclic R -module is multiplication, we get the following result.

Corollary 2.7. if M is a cyclic R -module, then M is prime if and only if M is an essentially quasi-prime.

Recall that an R -module M is uniform, if every nonzero submodule of M is essential in M ⁽⁹⁾.

Proposition 2.8. Let M a uniform R -module. Then M is essentially quasi-prime R -module if and only if M is prime.

Proof. Suppose that M is essentially quasi-prime R -module, we must prove that $\text{ann}_R M = \text{ann}_R(m)$ for every $0 \neq m \in M$. Let $x \in \text{ann}_R(m)$ then $x(m) = (0)$. Let m' be a non zero element of M . Since M is uniform then $(m) \cap (m') \neq (0)$. That is there exists $r_1, r_2 \in R$ such that $r_1 m = r_2 m' \neq 0$. But $xm = 0$, implies that $xr_1 m = 0$. It follows that $xr_1 m = xr_2 m' = 0$, hence $x \in \text{ann}_R(r_2 m')$. But M is essentially quasi-prime R -module, then by corollary 2.3, $x \in \text{ann}_R(m')$. That is $xm' = (0)$ for each $0 \neq m' \in M$. Thus, $x \in \text{ann}_R(M)$ and hence $\text{ann}_R M = \text{ann}_R(m)$. Therefore M is prime.

The converse is direct

Recall that an R -module M is quasi-Dedekind, if

$\text{Hom}\left(\frac{M}{N}, M\right) = (0)$ for all non-zero submodule N of M (Mijbass A. S.)⁽¹⁰⁾.

Proposition 2.9. Every quasi-Dedekind R -module is an essentially quasi-prime module.

3. Essential Pseudo-prime Modules.

In this section, we introduce the definition of essentially pseudo-prime R -modules as generalization of prime modules.

Before we introduce the definition of an essentially pseudo-prime R -module, we need to introduce the definition of essentially pseudo-prime submodule.

Definition 3.1. A proper submodule N of an R -module M is called essentially pseudo-prime if $[N, K]$ is a primary ideal of R for each non-zero essential submodule K of M such that $N \subset K$.

It is clear that every prime submodule is essentially pseudo-prime.

The following result is a characterization of essentially pseudo-prime submodule.

Proposition 3.2. Let M be an R -module, and N be a proper

submodule of M . Then N is an essentially pseudo-prime submodule of M if and only if $\sqrt{[N: K]} = \sqrt{[N: rK]}$ for each non-zero essential submodule K of M such that $N \subset K, rK \not\subseteq N, r \in R$.

Proof. Suppose that N is essentially pseudo-prime submodule, it is clear that $\sqrt{[N: K]} \subseteq \sqrt{[N: rK]}$ for each non-zero essentially submodule K of M with $N \subset K, rK \not\subseteq N, r \in R$. Let $a \in \sqrt{[N: rK]}$, then $a^n \in \sqrt{[N: rK]}$ for some $n \in \mathbb{Z}^+$ and then $a^n rK \subseteq N$, implies that $a^n r \in [N: K]$. But $[N: K]$ is primary ideal of R and $r \notin [N: K]$, so $(a^n)^l \in [N: K]$ for some $l \in \mathbb{Z}^+$. Then $a \in [N: K]$. Therefore $\sqrt{[N: rK]} \subseteq \sqrt{[N: K]}$, and hence $\sqrt{[N: K]} = \sqrt{[N: rK]}$.

Conversely : To prove $[N: K]$ is primary ideal for each non-zero essential submodule K of M with $N \subset K$. Let $ar \in [N: K]$ and suppose that $r \notin [N: K]$. Then $a \in [N: rK] \subseteq \sqrt{[N: rK]} = \sqrt{[N: K]}$. Thus $a \in \sqrt{[N: K]}$, and hence N is an essentially pseudo-prime submodule of M .

Proposition 3.3 Let M be a multiplication R -module and N be a proper submodule of M .

Then the following statements are equivalents:

1. N is essentially pseudo-prime submodule of M .
2. $[N:M]$ is a primary ideal of R .
3. N is a primary submodule of M .

Proof. $1 \Rightarrow 2$ It follows directly.

$2 \Rightarrow 3$ Since M is multiplication module, then by [Remark, 2.15]⁽²⁾ $[K:M] \not\subseteq [N:M]$ for each submodule K of M such that $N \subset K$. To prove that N is primary submodule of M , let $r \in R, x \in M$ such that $rx \in N$ and suppose that $x \notin N$. It is clear that the submodule $K = N + (x) \supset N$ and so $[K:M] \not\subseteq [N:M]$. Then there exist $s \in [K:M]$ and $s \notin [N:M]$. Thus $sM \subseteq K$ and $sM \not\subseteq N$. But $sM \subseteq K$, implies $rsM \subseteq rK = r(N + (x)) \subseteq N$ and $rs \in [N:M]$. But $[N:M]$ is primary ideal of R and $s \notin [N:M]$, $r^n \in [N:M]$ for some $n \in \mathbb{Z}^+$. Therefore N is primary submodule of M .

$3 \Rightarrow 1$ Directly.

Proposition 3.4. Let M be an R -module, and N be a proper submodule of M . Then N is essentially pseudo-prime submodule of M if and only if

$[N:I]$ is an essentially pseudo-prime submodule of M .

Proof. To prove that $[N:I]$ is essentially pseudo-prime submodule of M , we must prove that $[[N:I]:K]$ is a primary ideal of R for each a non-zero essential submodule K of M such that $[N:I] \subseteq K$. Note that $N \subseteq [N:I] \subseteq K$. Let $a, b \in R$ such that $ab \in [[N:I]:K]$, and suppose $a \notin [[N:I]:K]$, then $abKI \subseteq N$ and $aKI \not\subseteq N$.

Hence $abI \subseteq N$. But $[N:K]$ is primary ideal and $aI \not\subseteq [N:K]$. Then $b^n \in [N:K]$ for some $n \in \mathbb{Z}^+$, and so $b^n K \subseteq N \subseteq [N:I]$. Therefore $b^n \in [[N:I]:K]$ and hence $[N:I]$ is an essentially pseudo-prime submodule of M .

The converse follows by taking $I = R$.

The following proposition studied the homomorphic image and the inverse image of essentially pseudo-prime submodule.

Proposition 3.5. Let M and M' be two R -modules, and $f: M \rightarrow M'$ be an epimorphism, then:

- 1) If N is essentially pseudo-prime submodule of M' , then $f^{-1}(N)$ is an

essentially pseudo-prime submodule of M .

- 2) If N is an essentially pseudo-prime submodule of M with $\text{Ker } f \subseteq N$, then $f(N)$ is an essentially pseudo-prime submodule of M' .

Proof. (1) To prove that $f^{-1}(N)$ is an essentially pseudo-prime submodule of M , we must prove that $[f^{-1}(N):K]$ is a primary ideal of R for each non-zero essential submodule K of M such that $f^{-1}(N) \subseteq K$. Since f is an epimorphism, then it is clear that $f(K)$ is essential submodule of M' . Now, let $a, b \in R$ such that $a, b \in [f^{-1}(N):K]$, then $abK \subseteq f^{-1}(N)$. Hence $abf(K) \subseteq f(f^{-1}(N))$, it follows that $abf(K) \subseteq f(f^{-1}(N)) = N$.

Hence $ab \in [N:f(K)]$, but $f^{-1}(N) \subseteq K$, implies that $N \subseteq f(K)$ and hence $[N:f(K)]$ is a primary ideal of R . Therefore either $a \in [N:f(K)]$ or $b^n \in [N:f(K)]$ for some $n \in \mathbb{Z}^+$. And so, either $af(K) \subseteq N$ or $b^n f(K) \subseteq N$. Thus either $f(aK) \subseteq N$ or $f(b^n K) \subseteq N$. Hence $aK \subseteq f^{-1}(N)$ or $b^n K \subseteq f^{-1}(N)$. It follows that $a \in [f^{-1}(N):K]$ or $b^n \in [f^{-1}(N):K]$. Thus $[f^{-1}(N):K]$ is a primary ideal of R . Hence

$f^{-1}(N)$ is an essentially pseudo-prime submodule of M .

(2) To prove that $f(N)$ is an essentially pseudo-prime submodule of M' , we must prove that $[f(N):K']$ is a primary ideal of R for each non-zero essential submodule K' of M' such that $f(N) \subseteq K'$. Since K' is essential in M' , then by (Kash F.)⁽⁷⁾ $f^{-1}(K')$ is essential in M . since f is epimorphism, then $K' = f(f^{-1}(K'))$. Put $K = f^{-1}(K')$. It follows that $f(N) \subseteq f(K)$. So to prove that $[f(N):f(K)]$ is a primary ideal of R . Let $a, b \in R$ such that $a, b \in [f(N):f(K)]$ so $abf(K) \subseteq f(N)$. Hence for each $abf(x) \in f(N)$ for each x in N , which implies that $abx - n \in \text{Ker } f$ and so $abx \in N$ for each x in K . Hence $ab \in [N:K]$. But $[N:K]$ is primary ideal of R , so either $a \in [N:K]$ or $b^n \in [N:K]$ for some $n \in \mathbb{Z}^+$. Thus either $aK \subseteq N$ or $b^n K \subseteq N$. and so either $af(K) \subseteq f(N)$ or $b^n f(K) \subseteq f(N)$. That is either $a \in [f(N):f(K)]$ or $b^n \in [f(N):f(K)]$ which mean that $[f(N):f(K)]$ is a primary ideal of R . Hence $f(N)$ is an essentially pseudo-prime submodule of M' .

Now we introduce the definition of an essentially pseudo-prime module.

Definition 3.6. An R -module M is said to be an essentially pseudo-prime module if $\text{ann}_R N$ is primary ideal of R for each non-zero essential submodule N of M .

It is clear that every prime R -module is an essentially pseudo-prime module, but the converse is not true, since the Z -module Z_4 is an essentially pseudo-prime Z -module, but not prime Z -module. Also Z_{p^∞} as a Z -module is an essentially pseudo-prime Z -module but not prime Z -module. It is clear that every an essentially quasi-prime module is an essentially pseudo-prime module, but the converse is not true, since Z_{p^∞} as a Z -module is an essentially pseudo-prime Z -module, but not an essentially quasi-prime module Z - module. The homomorphic of an essentially pseudo-prime module need not be an essentially pseudo-prime module, for example, it is clear that the Z -module Z is an essentially pseudo-prime module, but $\frac{Z}{6Z} \cong Z_6$ is not an essentially pseudo-prime Z -module because the only essential submodule of

is Z_6 is Z_6 itself, and $\text{ann}_Z Z_6 = 6Z$ which is not primary ideal of Z .

The following result is a characterization of an essentially pseudo-prime module.

Proposition 3.7. Let M be an R -module, then M is an essentially pseudo-prime R -module if and only if (0) is an essentially pseudo-prime submodule.

Proof. Suppose that M is an essentially pseudo-prime module, to prove that (0) is an essential pseudo-prime submodule of M . Since M is an essentially pseudo-prime module, then $\text{ann}_R K$ is primary ideal of R for each non-zero essential submodule K of M . But $\text{ann}_R K = [(0):K]$ is a primary ideal for each non-zero essential submodule containing (0) . Hence (0) is an essentially pseudo-prime submodule of M .

Conversely; Since (0) is an essentially pseudo-prime submodule of M , then $[(0):\text{ann}_R K]$ is primary ideal of R for each non-zero essential submodule K of M contained (0) .

But $[(0):\text{ann}_R K] = \text{ann}_R K$. So, M is an essentially pseudo-prime module.

Now, we can easily obtain the following corollary.

Corollary 3.8. A proper submodule N of an R -module M is an essentially pseudo-prime if and only if $\frac{M}{N}$ is an essentially pseudo-prime module.

By using proposition 3.7 and proposition 3.4 we get the following corollary.

Corollary 3.9. An R -module M is an essentially pseudo-prime module if and only if $\text{ann}_M I$ is an essentially pseudo-prime submodule of M for each non-zero ideal I of R .

By using proposition 3.2 and proposition 3.7 we get the following characterization for an essentially pseudo-prime module.

Proposition 3.10. Let M be an R -module. Then M is an essentially pseudo-prime module if and only if $\sqrt{\text{ann}_R K} = \sqrt{\text{ann}_R rK}$ for each non-zero essential submodule K of M such that $rK \neq (0)$ $r \in R$.

As a direct consequence of proposition 3.10, we get the following result.

Corollary 3.11. Let M be an R -module, and M is an essentially pseudo-prime module, then

$\sqrt{\text{ann}_R(m)} = \sqrt{\text{ann}_R(rm)}$ for each non-zero essential submodule (m) of M generated by $0 \neq m \in M$ with $r(m) \neq (0)$, $r \in R$.

The following proposition shows that the two concepts prime module and an essentially pseudo-prime module are equivalent under the certain condition.

Proposition 3.12. Let M be a uniform R -module and $\text{ann}_R N$ is a semi-prime ideal of R for each non-zero submodule of M . then M is prime if and only if M is an essentially pseudo-prime module.

Proof. To prove that M is a prime, we must prove (0) is a prime submodule. Let $rx = 0$, for $r \in R, x \in M$ and suppose that $x \neq 0$, since M is uniform, then $(x) \cap (y) \neq (0) \forall y \in M, y \neq 0$, and so, there exists a non zero elements $a, b \in R$ such that $ax = ay \neq 0$. But $rx = 0$, so $rax = 0$. It follows that $rax = ray$, and so $r \in \text{ann}_R(by)$. Hence by corollary 3.11 $r \in \sqrt{\text{ann}_R(y)}$ for each y in M . On other hand $\text{ann}_R N$ is a semi-prime ideal of R , so $r \in \text{ann}_R(y)$ for all y in M , and so $r \in \text{ann}_R M$. Thus (0) is a

prime submodule of M . Hence M is prime module.

The converse is direct.

Proposition 3.13. Let M be a uniform R -module and $\text{ann}_R N$ is a semi-prime ideal of R for each non-zero submodule N of M . Then the following statements are equivalents:

1. M is an essentially pseudo-prime module.
2. M is prime module.
3. M is a quasi-Dedekind module.

Proof. $1 \Rightarrow 2$ by proposition 3.12.

$2 \Rightarrow 3$ by [Theo.3.11]⁽¹⁰⁾

$3 \Rightarrow 1$ by [Theo.1.7]⁽¹⁰⁾ we get M is prime, hence an essentially pseudo-prime module.

Corollary 3.14. Let M be a uniform R -module, and $\text{ann}_R N$ is semi-prime ideal of R for each non-zero submodule N of M . then the following statements are equivalents:

1. M is an essentially pseudo-prime module.
2. M is an essentially quasi-prime module.
3. M is a quasi-Dedekind module.

Proof. $1 \Rightarrow 2$ clearly.

$2 \Rightarrow 3$ by proposition 2.8, and [Theo.3.11]⁽¹⁰⁾.

$3 \Rightarrow 1$ by proposition 2.9.

4. Essentially Semi-Prime Modules.

In this section, we introduce the essentially semi-prime modules as a generalization of prime modules. We give in this section some characterizations, examples and properties of this concept.

Definition 4.1. An R -module M is called an essentially semi-prime module if $\text{ann}_R N$ is semi-prime ideal of R for each a non-zero essential submodule N of M .

It is clear that every prime module is an essentially semi-prime module, but the converse is not true, since the $\mathbb{Z}$ -module $\mathbb{Z}_6$ is essentially semi-prime $\mathbb{Z}$ -module but not prime because the only essential submodule of $\mathbb{Z}_6$ is $\mathbb{Z}_6$ itself and $\text{ann}_{\mathbb{Z}} \mathbb{Z}_6 = 6\mathbb{Z}$ is semi prime ideal of $\mathbb{Z}$. Also $\mathbb{Z}_6$ as $\mathbb{Z}$ -module is not essentially quasi-prime module and not essentially pseudo-prime module. $\mathbb{Z}_4$ as $\mathbb{Z}$ -module is an essentially pseudo-prime module, but not essentially semi-prime module. That is essentially pseudo-prime modules and essentially semi-prime module are independent.

Also the homomorphic image of an essentially semi-prime module is not an essentially semi-prime module, for example the Z -module Z is an essentially semi-prime module because it is a prime and $\frac{Z}{4Z} \cong Z_4$ is not essentially semi-prime Z -module.

The following theorem gives characterization of essentially semi-prime modules.

Theorem 4.2. Let M be an R -module. Then the following statements are equivalent:

1. M is an essentially semi-prime module.
2. $[ann_R N : A]$ is semi prime ideal of R for every non-zero essential submodule N of M and for every ideal A of R such that $ann_R N \subseteq A$.
3. $[ann_R N : (r)]$ is a semi prime ideal of R for every non-zero essential submodule N of M and every ideal of R generated by $r \in R, r \notin ann_R N$.
4. $ann_R(m)$ is a semi-prime ideal of R for every non-zero essential submodule of M generated by $m \in M, m \neq 0$.

Proof. (1) $\Rightarrow$ (2) Since M is an essentially semi-prime module,

then $ann_R N$ is semi prime ideal of R for every non-zero essential submodule N of M . Assume that A is an ideal of R such that $ann_R N \subseteq A$. Hence by [Prop.1.4(2)]⁽⁴⁾ follows that $[ann_R N : A]$ is a semi-prime ideal of R .

(2) $\Rightarrow$ (3) It follows by [Prop.1.4(3)]⁽⁴⁾.

(3) $\Rightarrow$ (4) Let (m) be a non-zero essential submodule of M generated by $m \in M, m \neq 0$. Since $1 \notin ann_R(m)$ and $[ann_R(m) : R] = ann_R(m)$ is a semi-prime ideal of R .

(4) $\Rightarrow$ (1) Let N be a non zero essential submodule of N , and $ann_R(m)$ is a semi-prime ideal for each non-zero essential submodule generated by $m \in M, m \neq 0$. Then $ann_R N = \bigcap_{x \in N} ann(x)$ is a semi-prime ideal of R . Hence M is an essentially semi-prime module.

Proposition 4.3. Let M be an R -module. Then M is an essentially semi-prime module if and only if (0) is a semi-prime submodule of M .

Proof. Since M is essentially semi-prime module, then $ann_R(m)$ is semi –prime ideal of R for each non-zero for each essential submodule generated

$m \in M, m \neq 0$. Now, let $r^2m \in (0)$ where $m \in M, m \neq 0, r \in R$. Thus $r^2m = 0$, then $r^2 \in \text{ann}_R(m)$, hence $r \in \text{ann}_R(m)$. That is $rm = 0$, thus $rm \in (0)$. Therefore (0) is semi-prime submodule of M .

Conversely; Let (m) be a non-zero essential submodule of M generated by $m \in M, m \neq 0$. Suppose that $r^2 \in \text{ann}_R(m)$, hence $r^2m = 0$ that is $sr^2m \in (0)$, but (0) is a semi-prime submodule of M , hence $rm \in (0)$. That is $rm = 0$, hence $r \in \text{ann}_R(m)$. Therefore $\text{ann}_R(m)$ is a semi-prime ideal of R .

Recall that an R -module M is a Z -regular if for all $m \in M$, there exists $f \in \text{hom}(M, R) = M^*$ such that $f(m)m = m$ (Zelmanowitz J.)⁽¹¹⁾.

Proposition 4.4. If M is Z -regular R -module, then M is essentially semi-prime module.

Proof. Assume that (m) is a non-zero essential submodule of M generated by $m \in M, m \neq 0$. Let $r^2 \in \text{ann}_R(m)$. Since M is a Z -regular, then there exists $f \in M^*$ such that $(rm)rm = rm$. Thus $f(m)r^2m = rm$. But $r^2 \in \text{ann}_R(m)$, thus $r^2m = 0$, hence $rm = 0$, whence $r \in \text{ann}_R(m)$. Therefore

$\text{ann}_R(m)$ is semi-prime. Hence by proposition 4.2 M is an essentially semi-prime module.

The following proposition shows that in the class of uniform modules, the two concepts prime modules and essentially semi-prime module are equivalent.

Proposition 4.5. Let M be a uniform R -module. Then M is an essentially semi-prime module if and only if M is a prime.

Proof. Suppose that M is essentially semi-prime module, and N a non-zero submodule of M . To prove that $\text{ann}_R M = \text{ann}_R N$. It is clear that $\text{ann}_R M \subseteq \text{ann}_R N$. Let $s \in \text{ann}_R N$ and $s \notin \text{ann}_R M$. Then, there exists $m \in M, m \neq 0$ such that $sm \neq 0$. But M is uniform, then $N \cap (sm) \neq (0)$, thus there exists a non zero element $k \in N \cap (sm)$. Hence $k = rsm$, where $r \in R$. Thus $sk = rs^2m = 0$. Hence $s^2 \in \text{ann}_R(rm)$, and $rm \neq 0$, so $\text{ann}_R(rm)$ is semi prime ideal of R , therefore $s \in \text{ann}_R(rm)$ and this implies $k = rsm = 0$ a contradiction. Thus $\text{ann}_R N = \text{ann}_R M$. Therefore M is a prime.

The converse is direct.

The following proposition shows that in the class of uniform module the concepts quasi-

Dedekind modules and essentially semi-prime module are equivalent.

Proposition 4.6. Let M be a uniform R -module. Then M is an essentially semi-prime if and only if M is a quasi-Dedekind.

Proof. Suppose that M is an essentially semi-prime module, hence by proposition 4.5 M is a prime. Then by [Theo.3.11]⁽¹⁰⁾ M is a quasi-Dedekind.

Conversely; by [Theo.1.7]⁽¹⁰⁾, we get M is a prime. Hence M is an essentially semi-prime module.

From propositions 4.5, 4.6, and 2.8 we get the following.

Corollary 4.7. Let M be a uniform R -module. Then the following statements are equivalent.

1. M is an essentially semi-prime module.
2. M is quasi-Dedekind module
3. M is an essentially quasi-prime module.
4. M is a prime.

References.

(1) Maghaderi J. and Nekooi R. “Strongly prime submodules

and pseudo Valuation Modules,” International Electronic J. of Algebra, 10 (2011), pp. 65-75.

(2) Lu, C.P. “Prime submodules of modules” Comment. Math., Univ. St. paula, Houston J. Math. 33 (1984) pp.61-69.

(3) Smith P. F. “primary Modules over commutative Rings” Glasgow J. Math. 43 (2001) pp.103-111.

(4) Eman A. A. (1996) “Prime and semi-prime submodules” M.Sc. Thesis, Baghdad Univ.

(5) Mohammedali I.H. and Gahwi T.Y. “Essentially prime modules and related modules” J. of Basrah 37 (2011), pp.187-196.

(6) Abdul-razak H.M. (1999) “Quasi-prime modules and Quasi-prime submodule” M.Sc. Thesis, Univ. of Baghdad.

(7) Kash F. “Modules and Rings” Academic Press, London, New York, (1976).

(8) Barnard A. “Multiplication modules” J. Algebra 71 (1981) pp.174-178.

(9) Goodreal K.R. (1982) “Ring Theory” Marcel-Dekker, New York Basel.

(10) Mijbass A. S. (1997) “Quasi Dedekind modules and Quasi invertible submodule”

Ph.D. Thesis, Univ. of (11) Zelman owitz J. “ Regular
Baghdad. modules” Trans. Amer. Math.
Soc., 163 (1972),341-355.

بعض اعمامات المقاسات الاولى

هبة كريم الركابي* و مؤمون فتاح العباسي**

*قسم الرياضيات, كلية علوم الحاسوب والرياضيات

**قسم الرياضيات, كلية التربية, جامعة تكريت

الخلاصة

المقاس M على الحلقة R يدعى مقاساً أولياً اذا كان $\text{ann}_R M = \text{ann}_R N$ لكل مقاس غير صفري جزئي N من M في هذا البحث قدمنا المقاسات الجوهرية الظاهرية الاولى، والمقاسات اللاجوهية الكاذبة الاولى والمقاسات الجوهرية الشبه الاولى كأعمامات للمقاسات الاولى ودرسنا العلاقة بينهما. العديد من التشخيصات والخواص والامثلة اعطيت. بالاضافة الى ذلك درسنا علاقة هذه الاعمامات مع بعض أصناف المقاسات الاخرى.