

Simultaneous Approximation By Generalization of Szász Type Operators

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Abstract

In the present paper, we introduce a generalization of the Summation-Integral Szász type operators denoted by $\tilde{S}_{n,p}(f(t); x)$. First, we prove the convergence theorem for the operator. Then, we find a recurrence relation of the m -th order moment for the operator $\tilde{S}_{n,p}(f(t); x)$. Finally, we give a Voronovskaja-type asymptotic formula and an error estimate in terms of modulus of continuity of the function being approximate.

Keywords and phrases: Linear positive operator, Simultaneous approximation, Voronovskaja-type asymptotic formula, Degree of approximation, Modulus of continuity.

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1. Introduction:

The Szász-Mirakyan operators are defined by

$$L_n(f; x) = \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right),$$

$$x \in R_0 := [0, \infty), n \in N := \{1, 2, \dots\}$$

where

$$q_{n,k}(x) = \frac{(nx)^k}{e^{nx} k!}, k \in N^0 := N \cup \{0\}$$

The operators $L_n(f; x)$ knew by the classical Szász operators (Szász). After that, Kasana et. al. (H. S. Kasana et al), proposed a modification of the classical Szász operators to approximate a space of functions integrable on R_0 as:

$$R_n(f(t); x) = n \sum_{k=0}^{\infty} q_{n,k}(x) \int_0^{\infty} q_{n,k}(t) f(t) dt.$$

Rempulska and Graczyk in the paper [3] introduced a modification of the Szász operators and studied some direct results in ordinary approximation as:

$$M_{n,r}(f, x) = \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} f\left(\frac{rk}{n}\right), x \in R_0, n \in N,$$

$$\text{where } A_r(t) = \sum_{k=0}^{\infty} \frac{t^{rk}}{(rk)!} \text{ for } t \in R_0.$$

In this paper, we introduced a new sequence of linear positive operators $\tilde{S}_{n,p}(f(t); x)$ given as follows:

$$\begin{aligned} & \tilde{S}_{n,p}(f(t); x) \\ &= \frac{1}{G_x} \left(n \sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t) f(t) dt \right. \\ & \left. + f(0) q_{n,0}(x) \right) \end{aligned} \quad (1)$$

2. Preliminaries :

We give some lemmas which help us in the proofs of main theorems.

Lemma 1: [2]

For $n, r \in N, x \in R_0$ and

$$A_r^{(m)}(nx) = n^m \sum_{k=m}^{\infty} \frac{(nx)^{rk-m}}{(rk-m)!} ; rk \geq m,$$

where $A_r(nx) = \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!}$, we get :

$$(1) \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} rk = x A_r'(nx);$$

$$(2) \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} (rk)^2 = x^2 A_r''(nx) + x A_r'(nx);$$

$$(3) \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} (rk)^3 = x^3 A_r'''(nx) + 3x^2 A_r''(nx) + x A_r'(nx);$$

$$\begin{aligned} (4) \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} (rk)^m &= x^m A_r^{(m)}(nx) \\ &+ \frac{m(m-1)}{2} x^{m-1} A_r^{(m-1)}(nx) + \\ &\text{terms of form} \\ &C x^l A_r^{(l)}(nx), \text{ where } 0 < l < m-1. \end{aligned}$$

Lemma 2: [3]

For $n, r \in N$ and $x \in (0, \infty)$, we get:

$$\frac{A_r^{(m)}(nx)}{n^m A_r(nx)} \rightarrow 1, \frac{A_r^{(m)}(nx)}{n^s A_r(nx)} = o(1)$$

as $n \rightarrow \infty, s = r + m, r \geq 1$.

$$\text{And } \frac{A_r^{(m)}(nx)}{n^s A_r(nx)} = o(n^{-r}).$$

Lemma 3: [1]

There exist polynomials $Q_{i,j,r}(x)$ independent of n and k for sufficiently large n , such that:

$$x^r q_{n,k}^{(r)}(x) = \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i (k - nx)^j Q_{i,j,r}(x) q_{n,k}(x).$$

3. The Convergence Theorem of

$$\tilde{S}_{n,p}(f(t); x):$$

We show that the operator $\tilde{S}_{n,p}(f; x)$ converges to $f(x)$ in $C_h[0, \infty)$ by applying the conditions of Korovkin's theorem.

Where $C_h[0, \infty)$ the space of all real valued continuous functions on interval $[0, \infty)$ such that $|f(t)| \leq C e^{ht}$ for some constant $C > 0$ and $h > 0, t \geq 0$.

Theorem 1:

For $n, p \in N$ and $x \in R_0$, the following conditions are hold:

$$(1) \quad \tilde{S}_{n,p}(1; x) = 1;$$

$$(2) \quad \tilde{S}_{n,p}(t; x) = \frac{x A'_p(nx)}{n A_p(nx)} \rightarrow x \text{ as } n \rightarrow \infty;$$

$$(3) \quad \tilde{S}_{n,p}(t^2; x) = \frac{x^2 A''_p(nx)}{n^2 A_p(nx)} + \frac{2x A'_p(nx)}{n^2 A_p(nx)} \rightarrow x^2 \text{ as } n \rightarrow \infty.$$

Therefore,

$$\tilde{S}_{n,p}(f(t); x) \rightarrow f(x) \text{ as } n \rightarrow \infty \text{ for all}$$

$$f \in C_h[0, \infty).$$

Proof:

Using **Lemma's 1** and **2** the consequence (1) is easily verified, hence the details are omitted.

The proof of (2) is given below:

$$\begin{aligned} \tilde{S}_{n,p}(t; x) &= \frac{1}{G_x} \left(n \sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t) t dt \right. \\ &\quad \left. + 0 \right) \\ &= \frac{1}{n G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) (pk) \\ &= \frac{x A'_p(nx)}{n A_p(nx)} \quad (\text{in view of Lemma 2}) \end{aligned}$$

$$\rightarrow x \text{ as } n \rightarrow \infty. \quad (\text{in view of Lemma 2})$$

Using the same technique and **Lemma's 1** and **2** the value of $\tilde{S}_{n,p}(t^2; x)$ is followed immediately as:

$$\begin{aligned} \tilde{S}_{n,p}(t^2; x) &= \frac{x^2 A''_p(nx)}{n^2 A_p(nx)} + \frac{2x A'_p(nx)}{n^2 A_p(nx)} \\ &\rightarrow x^2 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore, by Korovkin's, we get:

$$\tilde{S}_{n,p}(f(t); x) \rightarrow f(x) \text{ as } n \rightarrow \infty. \quad \blacksquare$$

4. The m -th Order Moment for

$$\tilde{S}_{n,p}(f(t); x):$$

In this part, we give the definition of the m -th order moment for the operators $\tilde{S}_{n,p}(f; x)$ and prove a recurrence relation for this moment.

Definition 1:

The m -th order moment ($m \in N^0$) for the operators $\tilde{S}_{n,p}(f(t); x)$ is defined by:

$$\begin{aligned} \tilde{T}_{n,m}(x) &= \tilde{S}_{n,p}((t-x)^m; x) \\ &= \frac{1}{G_x} \left(n \sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t) (t \right. \\ &\quad \left. - x)^m dt + (-x)^m e^{-nx} \right) \\ &= U_{n,m}(x) \\ &\quad + \frac{(-x)^m e^{-nx}}{G_x}; \end{aligned} \quad (2)$$

Lemma 4:

For the function $U_{n,m}(x)$ which is defined above, we have:

$$\begin{aligned} U_{n,0}(x) &= 1 - e^{-nx}, U_{n,1}(x) \\ &= x \left(\frac{G'_x}{nG_x} + e^{-nx} \right), \\ U_{n,2}(x) &= \frac{x^2 G''_x + 2x G'_x}{n^2 G_x} - \frac{2x}{n} - x^2 e^{-nx}, \end{aligned}$$

and

$$\begin{aligned} nU_{n,m+1}(x) &= x \left\{ U'_{n,m}(x) \right. \\ &\quad \left. + 2mU_{n,m-1}(x) \right. \\ &\quad \left. + \frac{G'_x}{G_x} U_{n,m}(x) \right\} \\ &\quad + mU_{n,m}(x), \quad m \geq 1, \end{aligned}$$

where $G_x = e^{-nx} A_p(nx)$ and $G'_x = e^{-nx} (A'_p(nx) - nA_p(nx))$.

Proof:

By direct computation, we easily have the values of $U_{n,0}(x)$, $U_{n,1}(x)$ and $U_{n,2}(x)$.

Now,

$$\begin{aligned} U'_{n,m}(x) &= \frac{n}{G_x^2} \left\{ G_x \left(\sum_{k=1}^{\infty} q'_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t) (t \right. \right. \\ &\quad \left. \left. - x)^m dt \right. \right. \\ &\quad \left. \left. - m \sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t) (t \right. \right. \\ &\quad \left. \left. - x)^{m-1} dt \right) \right. \\ &\quad \left. - G'_x \sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t) (t \right. \right. \\ &\quad \left. \left. - x)^m dt \right\}. \end{aligned}$$

Then

$$\begin{aligned} U'_{n,m}(x) + mU_{n,m-1}(x) + \frac{G'_x}{G_x} U_{n,m}(x) &= \frac{n}{G_x} \sum_{k=1}^{\infty} q'_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t) (t \\ &\quad - x)^m dt. \end{aligned}$$

Since $xq'_{n,pk}(x) = (pk - nt + n(t - x))q_{n,pk}(x)$ we get:

$$\begin{aligned} x \left\{ U'_{n,m}(x) + mU_{n,m-1}(x) + \frac{G'_x}{G_x} U_{n,m}(x) \right\} &= \frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t) (pk \\ &\quad - nt)(t - x)^m dt \\ &\quad + nU_{n,m+1}(x). \end{aligned}$$

$$= \frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t) ((pk-1)$$

$$-nt)(t-x)^m dt$$

$$+ \frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t)(t$$

$$-x)^m dt + nU_{n,m+1}(x)$$

we obtain:

$$x \left\{ U'_{n,m}(x) + mU_{n,m-1}(x) + \frac{G'_x}{G_x} U_{n,m}(x) \right\}$$

$$- nU_{n,m+1}(x) - U_{n,m}(x)$$

$$= \frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} tq'_{n,pk-1}(t)(t$$

$$-x)^m dt.$$

Using the identity $t = (t-x) + x$ and integration by parts, we have:

$$nU_{n,m+1}(x) = x \left\{ U'_{n,m}(x) \right.$$

$$+ 2mU_{n,m-1}(x)$$

$$\left. + \frac{G'_x}{G_x} U_{n,m}(x) \right\}$$

$$+ mU_{n,m}(x); \quad m \geq 1. \blacksquare$$

Lemma 5:

For the function $\tilde{T}_{n,m}(x)$ which is defined in (2), we have

$$\tilde{T}_{n,0}(x) = 1, \quad \tilde{T}_{n,1}(x)$$

$$= \frac{xG'_x}{nG_x} \quad \text{and} \quad \tilde{T}_{n,2}(x)$$

$$= \frac{x^2G''_x}{n^2G_x} + \frac{2xG'_x}{n^2G_x} + \frac{2x}{n}.$$

The recurrence relation is:

$$n\tilde{T}_{n,m+1}(x) = x \left\{ \tilde{T}'_{n,m}(x) + 2m\tilde{T}_{n,m-1}(x) \right.$$

$$\left. + \frac{G'_x}{G_x} \tilde{T}_{n,m}(x) \right\}$$

$$+ m\tilde{T}_{n,m}(x), m \geq 1. \quad (3)$$

where $G_x = e^{-nx}A_p(nx)$ and $G'_x = e^{-nx}(A'_p(nx) - nA_p(nx))$.

Further, we can determine for sufficiently large n the following:

(i) $\tilde{T}_{n,m}(x)$ as a polynomial in x of degree m .

(ii) for every $x \in R_0$, $\tilde{T}_{n,m}(x) = O\left(n^{-\left[\frac{(m+1)}{2}\right]}\right)$.

Proof:

It is easy to show that

$$\tilde{T}_{n,0}(x) = 1, \quad \tilde{T}_{n,1}(x)$$

$$= \frac{xG'_x}{nG_x} \quad \text{and} \quad \tilde{T}_{n,2}(x)$$

$$= \frac{x^2G''_x}{n^2G_x} + \frac{2xG'_x}{n^2G_x} + \frac{2x}{n}.$$

From (2) and **Lemma 4**, we have:

$$\begin{aligned}
n\tilde{T}_{n,m+1}(x) &= n \left\{ U_{n,m+1}(x) + \frac{(-x)^{m+1}e^{-nx}}{G_x} \right\} \\
&= x \left\{ U'_{n,m}(x) + 2mU_{n,m-1}(x) + \frac{G'_x}{G_x} U_{n,m}(x) \right\} \\
&\quad + mU_{n,m}(x) + \frac{n(-x)^{m+1}e^{-nx}}{G_x} \\
&= x \left\{ \tilde{T}'_{n,m}(x) + \frac{n(-x)^m e^{-nx} + m(-x)^{m-1} e^{-nx}}{G_x} \right. \\
&\quad \left. + \frac{G'_x(-x)^m e^{-nx}}{G_x^2} \right\} + \\
&\quad 2mx \left\{ \tilde{T}_{n,m-1}(x) - \frac{(-x)^{m-1} e^{-nx}}{G_x} \right. \\
&\quad \left. + x \frac{G'_x}{G_x} \left\{ \tilde{T}_{n,m}(x) - \frac{(-x)^m e^{-nx}}{G_x} \right\} + \right. \\
&\quad \left. m \left\{ \tilde{T}_{n,m}(x) - \frac{(-x)^m e^{-nx}}{G_x} \right\} + \frac{n(-x)^{m+1} e^{-nx}}{G_x}, \right.
\end{aligned}$$

then

$$\begin{aligned}
&n\tilde{T}_{n,m+1}(x) \\
&= x \left\{ \tilde{T}'_{n,m}(x) + 2m\tilde{T}_{n,m-1}(x) + \frac{G'_x}{G_x} \tilde{T}_{n,m}(x) \right\} \\
&\quad + m\tilde{T}_{n,m}(x), \quad m \geq 1.
\end{aligned}$$

From the values of $\tilde{T}_{n,0}(x)$ and $\tilde{T}_{n,1}(x)$, it is clear that the consequences (i) and (ii) are hold for $m = 0$ and $m = 1$. The consequence (i) can be proved easily by using (3) and the induction on m .

Using the same technique in **Lemma 1.5** (ii) [2], the proof of consequence (ii) is holds.

■

Lemma 6:

For $m \geq 1$, we have:

$$\begin{aligned}
&\tilde{S}_{n,p}(t^m; x) \\
&= \frac{1}{n^m A_p(nx)} \left(x^m A_p^{(m)}(nx) \right. \\
&\quad \left. + m(m-1)x^{m-1} A_p^{(m-1)}(nx) \right. \\
&\quad \left. + \text{terms of form } C x^l A_p^{(l)}(nx), \quad l \right. \\
&\quad \left. < m-1 \right)
\end{aligned}$$

Proof:

By direct computation, **Lemma 1** and using (1.1) [2], we get:

$$\begin{aligned}
 & \tilde{S}_{n,p}(t^m; x) \\
 &= \frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t) t^m dt \\
 &+ 0 \\
 &= \frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \frac{((pk-1)+m)!}{(pk-1)! n^{m+1}} \\
 &= \frac{1}{n^m G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \left((pk-1) \right. \\
 &\quad \left. + m)((pk-1)+m \right. \\
 &\quad \left. - 1) \dots pk \right) \\
 &= \frac{1}{n^m G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \left((pk)^m \right. \\
 &\quad \left. + \frac{m(m-1)}{2} (pk)^{m-1} \right. \\
 &\quad \left. + \text{T. L. P. } (pk) \right)
 \end{aligned}$$

Where T. L. P. the terms in lower power of x .

$$\begin{aligned}
 &= \frac{1}{n^m A_p(nx)} \left(x^m A_p^{(m)}(nx) \right. \\
 &\quad \left. + \frac{m(m-1)}{2} x^{m-1} A_p^{(m-1)}(nx) \right. \\
 &\quad \left. + \frac{m(m-1)}{2} x^{m-1} A_p^{(m-1)}(nx) \right. \\
 &\quad \left. + \text{terms of form } C x^l A_p^l(nx), \quad l \right. \\
 &\quad \left. < m-1 \right) \\
 &= \frac{1}{n^m A_p(nx)} \left(x^m A_p^{(m)}(nx) \right. \\
 &\quad \left. + m(m-1)x^{m-1} A_p^{(m-1)}(nx) \right. \\
 &\quad \left. + \text{terms of form } C x^l A_p^{(l)}(nx), \quad l \right. \\
 &\quad \left. < m-1 \right). \quad \blacksquare
 \end{aligned}$$

Now, we prove that the derivatives of the operator (1) are approximation processes for corresponding order derivatives of the function being approximated.

Theorem 2:

Suppose that $r \in N, f \in C_h[0, \infty)$ for some $h > 0$ and $f^{(r)}$ exists at a point $x \in (0, \infty)$, then

$$\lim_{n \rightarrow \infty} \tilde{S}_{n,p}^{(r)}(f(t); x) = f^{(r)}(x). \quad (4)$$

Further, if $f^{(r)}$ exists and is continuous on $(a - \eta, b + \eta) \subset (0, \infty)$, $\eta > 0$, then (4) holds uniformly on $[a, b]$.

Proof:

By using Taylor's expansion of f , we get

$$\begin{aligned} f(t) &= \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} (t-x)^i \\ &\quad + \varepsilon(t, x)(t-x)^r; \quad \varepsilon(t, x) \\ &\rightarrow 0 \text{ as } t \rightarrow x. \end{aligned}$$

Hence,

$$\begin{aligned} \tilde{S}_{n,p}^{(r)}(f(t); x) &= \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} \tilde{S}_{n,p}^{(r)}((t-x)^i; x) \\ &\quad + \tilde{S}_{n,p}^{(r)}(\varepsilon(t, x)(t-x)^r; x) \\ &:= \Sigma_1 + \Sigma_2. \end{aligned}$$

By using **Lemma 6**, if $i < r$ we have

$$\tilde{S}_{n,p}^{(r)}(t^i; x) \rightarrow 0.$$

$$\Sigma_1 = \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} \tilde{S}_{n,p}^{(r)}((t-x)^i; x)$$

$$\begin{aligned} &= \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} \sum_{j=0}^r \binom{r}{j} (-x)^{r-j} \tilde{S}_{n,p}^{(r)}(t^j; x) \\ &\rightarrow \frac{f^{(r)}(x)}{r!} \tilde{S}_{n,p}^{(r)}(t^r; x) \\ &\rightarrow f^{(r)}(x) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Σ_2

$$\begin{aligned} &= \frac{d^r}{dx^r} \left(\frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t) \varepsilon(t, x) (t-x)^r dt + \frac{1}{G_x} \varepsilon(0, x) (-x)^r e^{-nx} \right); \\ &= I_1 + I_2. \end{aligned}$$

Next, making use of **Lemma 3** and replace k by pk and the fact

$$\frac{1}{G_x} \geq \frac{1}{|G_x^{(r)}|}, \quad r \in N^0, \text{ we have:}$$

$$\begin{aligned} |I_1| &\leq \frac{1}{G_x} \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} n^{i+1} \frac{|Q_{i,j,r}(x)|}{x^r} \sum_{k=1}^{\infty} q_{n,pk}(x) |pk - nx|^j \int_0^{\infty} q_{n,pk-1}(t) |\varepsilon(t, x)(t-x)^r| dt. \end{aligned}$$

Since $\varepsilon(t, x) \rightarrow 0$ as $t \rightarrow x$, for given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\varepsilon(t, x)| < \varepsilon$, whenever $0 < |t-x| < \delta$. For $|t-x| \geq \delta$, there exists a constant $C > 0$ such that $|\varepsilon(t, x)(t-x)^r| \leq Ce^{ht}$. Hence,

$$\begin{aligned}
 & |I_1| \\
 & \leq C \frac{n}{G_x} \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i \sum_{k=1}^{\infty} q_{n,pk}(x) |pk \\
 & - nx|^j \left(\varepsilon \int_{|t-x| < \delta} q_{n,pk-1}(t) |(t-x)^r| dt \right. \\
 & \left. + \int_{|t-x| \geq \delta} q_{n,pk-1}(t) |\varepsilon(t;x)(t-x)^r| dt \right) \\
 & := J_1 + J_2
 \end{aligned}$$

Since $\sup_{\substack{2i+j \leq r \\ i,j \geq 0}} \frac{|Q_{i,j,r}(x)|}{x^r} = M(x), x \in (0, \infty)$ but fixed.

Now, using Cauchy-Schwarz inequality for integration and then for summation, we are led to

$$\begin{aligned}
 J_1 & \leq C \frac{\varepsilon}{G_x} \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^{i+1} \sum_{k=1}^{\infty} q_{n,pk}(x) |pk \\
 & - nx|^j \left(\int_{|t-x| < \delta} q_{n,pk-1}(x) dt \right)^{\frac{1}{2}} \\
 & \times \left(\int_{|t-x| < \delta} q_{n,pk-1}(x) (t-x)^{2r} dt \right)^{\frac{1}{2}} \\
 & \leq C \varepsilon \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i \left(\frac{n^{2j}}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \left(\frac{pk}{n} \right. \right. \\
 & \left. \left. - x \right)^{2j} \right)^{\frac{1}{2}}
 \end{aligned}$$

$$\times \left(\frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \int_{|t-x| < \delta} q_{n,pk-1}(t) (t - x)^{2r} dt \right)^{\frac{1}{2}}.$$

We have:

$$\begin{aligned}
 & \frac{n^{2j}}{G_x} \left(\sum_{k=1}^{\infty} q_{n,pk}(x) \left(\frac{pk}{n} - x \right)^{2j} \right. \\
 & \left. + q_{n,0}(x)(-x)^{2j} - q_{n,0}(x)(-x)^{2j} \right) \\
 & = \frac{n^{2j}}{G_x} \left(\sum_{k=0}^{\infty} q_{n,pk}(x) \left(\frac{pk}{n} - x \right)^{2j} \right. \\
 & \left. - e^{-nx}(-x)^{2j} \right)
 \end{aligned}$$

$$= n^{2j} \left(O(n^{-j}) + O(n^{-s}) \right) = O(n^j) \quad \text{for any } s > 0,$$

Since,

$$\begin{aligned}
 & \frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \int_{|t-x| < \delta} q_{n,pk-1}(t) (t - x)^{2r} dt = U_{n,2r}(x) \\
 & = O(n^{-r})
 \end{aligned}$$

Hence,

$$\begin{aligned}
 J_1 & = C \varepsilon \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i O\left(n^{\frac{j}{2}}\right) O\left(n^{-\frac{r}{2}}\right) \\
 & = \varepsilon O\left(n^{\frac{2i+j}{2} - \frac{r}{2}}\right) = \varepsilon O(1).
 \end{aligned}$$

Next, again using Cauchy-Schwarz inequality for integration and then for summation, we have:

$$\begin{aligned}
& J_2 \\
& \leq \frac{C}{G_x} \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^{i+1} \sum_{k=1}^{\infty} q_{n,pk}(x) (pk \\
& \quad - nx)^j \left(\int_{|t-x| \geq \delta} q_{n,pk-1}(t) dt \right)^{\frac{1}{2}} \times \\
& \quad \left(\int_{|t-x| \geq \delta} q_{n,pk-1}(t) e^{2ht} dt \right)^{\frac{1}{2}} \\
& \leq C \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i \left(\frac{1}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) (pk - nx)^{2j} \right)^{\frac{1}{2}} \\
& \quad \times \left(\frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \int_{|t-x| \geq \delta} q_{n,pk-1}(t) e^{2ht} dt \right)^{\frac{1}{2}}
\end{aligned}$$

Making use of Taylor's expansion, Cauchy-Schwarz inequality for integration and then for summation and **Lemma 5**, we have:

$$I = O(n^{-s}); \quad s \geq 0,$$

where I

$$:= \frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) \int_{|t-x| \geq \delta} q_{n,pk-1}(t) e^{2ht} dt.$$

Therefore,

$$\begin{aligned}
& = C \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i O\left(n^{\frac{j}{2}}\right) O(n^{-s}) = o(1), \\
& \quad \text{for } s > \frac{r}{2}.
\end{aligned}$$

Now, since $\varepsilon > 0$ is arbitrary, it follows

that $I_1 \rightarrow 0$ as $n \rightarrow \infty$. Also $I_2 \rightarrow 0$

as $n \rightarrow \infty$ and hence $\Sigma_2 = o(1)$.

Combining the estimates Σ_1 and Σ_2 , (4) is immediate.

■

The next theorem is a Voronovskaja-type asymptotic formula for the operators $\tilde{S}_{n,p}^{(r)}(f(t); x)$, $r \in \mathbb{N}$.

Theorem 3:

Let $f \in C_h[0, \infty)$ for some $h > 0$. If $f^{(r+2)}$ exists at a point $x \in (0, \infty)$, then

$$\begin{aligned}
& \lim_{n \rightarrow \infty} n \left\{ \tilde{S}_{n,p}^{(r)}(f(t); x) - f^{(r)}(x) \right\} \\
& = r f^{(r+1)}(x) \\
& \quad + x f^{(r+2)}(x).
\end{aligned}$$

Further, if $f^{(r+2)}$ exists and is continuous on the interval $(a - \eta, b + \eta) \subset (0, \infty)$, $\eta > 0$, then (5) holds uniformly on $[a, b]$.

Proof:

By using Taylor's expansion of f , we have:

$$\begin{aligned}
f(t) &= \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} (t-x)^i \\
& \quad + \varepsilon(t, x) (t-x)^{r+2}; \quad \varepsilon(t, x) \\
& \quad \rightarrow 0 \text{ as } t \rightarrow x.
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{S}_{n,p}^{(r)}(f(t); x) &= \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \tilde{S}_{n,p}^{(r)}((t-x)^i; x) \\
& \quad + \tilde{S}_{n,p}^{(r)}(\varepsilon(t, x)(t-x)^{r+2}; x).
\end{aligned}$$

Using the same technique of **Theorem 2**,

we get:

$$\tilde{S}_{n,p}^{(r)}(\varepsilon(t, x)(t - x)^{r+2}; x) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then

$$\begin{aligned} \tilde{S}_{n,p}^{(r)}(f(t); x) &= \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \tilde{S}_{n,p}^{(r)}((t - x)^i; x) \\ &= \frac{f^{(r)}(x)}{r!} \tilde{S}_{n,p}^{(r)}(t^r; x) \\ &\quad + \frac{f^{(r+1)}(x)}{(r+1)!} ((-x)(r + 1) \tilde{S}_{n,p}^{(r)}(t^r; x) \\ &\quad + \tilde{S}_{n,p}^{(r)}(t^{r+1}; x) \\ &\quad + \frac{f^{(r+2)}(x)}{(r+2)!} \left(\frac{(r+1)(r+2)}{2} x^2 \tilde{S}_{n,p}^{(r)}(t^r; x) \right. \\ &\quad \left. + (r+2)(-x) \tilde{S}_{n,p}^{(r)}(t^{r+1}; x) + \tilde{S}_{n,p}^{(r)}(t^{r+2}; x) \right). \end{aligned}$$

For sufficiently large n and using **Lemma 2**, we get:

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left\{ \frac{f^{(r)}(x)}{r!} r! \right. \\ \left. + \frac{f^{(r+1)}(x)}{(r+1)!} \left((-x)(r + 1)r! + (r+1)!x \right. \right. \\ \left. \left. + \frac{r(r+1)r!}{n} \right) + \frac{f^{(r+2)}(x)}{(r+2)!} \right\} \end{aligned}$$

$$\begin{aligned} &\times \left(\frac{(r+1)(r+2)r!}{2} x^2 \right. \\ &\left. + (r+2)(-x) \left((r+1)!x + \frac{r(r+1)r!}{n} \right) \right. \\ &\left. + \frac{(r+2)!}{2} x^2 \right. \\ &\left. + \frac{(r+1)(r+2)(r+1)!}{n} x \right) - f^{(r)}(x) \Big\} \\ &= rf^{(r+1)}(x) + xf^{(r+2)}(x). \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left\{ \tilde{S}_{n,p}^{(r)}(f(t); x) - f^{(r)}(x) \right\} \\ = rf^{(r+1)}(x) \\ + xf^{(r+2)}(x). \quad \blacksquare \end{aligned}$$

Finally, we give an estimate of the degree of approximation by the operators $\tilde{S}_{n,p}^{(r)}(f(t); x)$.

Theorem 4:

Let $f \in C_h[0, \infty)$ for some $h > 0$ and $r \leq v \leq r+2$. If $f^{(v)}$ exists and continuous on $(a - \eta, b + \eta) \subset (0, \infty)$, $\eta > 0$, then for sufficiently large n ,

$$\begin{aligned} \left\| \tilde{S}_{n,p}^{(r)}(f(t); x) - f^{(r)}(x) \right\|_{C[a,b]} \\ \leq C_1 n^{-1} \sum_{i=r}^v \|f^{(i)}\|_{C[a,b]} \\ + C_2 n^{\frac{-1}{2}} \omega_{f^{(v)}} \left(n^{\frac{-1}{2}}; (a - \eta, b + \eta) \right) \\ + O(n^{-2}), \end{aligned}$$

where C_1, C_2 are constants independent of f and n , $\omega_f(\delta)$ is the modulus of continuity

of f on $(a - \eta, b + \eta)$, and

$\|\cdot\|_{C[a,b]}$ denotes the sup-norm on $[a, b]$.

proof:

By our hypothesis

$$f(t) = \sum_{i=0}^v \frac{f^{(i)}(x)}{i!} (t-x)^i + \frac{f^{(v)}(\xi) - f^{(v)}(x)}{v!} (t-x)^v \chi(t) + h(t, x)(1 - \chi(t)),$$

where ξ lies between t, x , and $\chi(t)$ is the characteristic function of the interval $(a - \eta, b + \eta)$.

For $t \in (a - \eta, b + \eta)$ and $x \in [a, b]$, we get:

By using **Lemma 6**, we get

$$\begin{aligned} \Sigma_1 &= \sum_{i=r}^v \frac{f^{(i)}(x)}{i!} \sum_{j=r}^i \binom{i}{j} (-x)^{i-j} \frac{d^r}{dx^r} \left(\frac{1}{n^j A_p(nx)} (x^j A_p^{(j)}(nx) \right. \\ &\quad \left. + j(j-1)x^{j-1} A_p^{(j-1)}(nx) + O(n^{-2}) \right) \\ &\quad - f^{(r)}(x). \end{aligned}$$

Consequently,

$$\begin{aligned} &\|\Sigma_1\|_{C[a,b]} \\ &\leq C_1 n^{-1} \left[\sum_{i=r}^v \|f^{(i)}\|_{C[a,b]} \right] \\ &\quad + O(n^{-2}), \text{ uniformly on } [a, b]. \end{aligned}$$

To estimate Σ_2 we proceed as follows:

$$f(t) = \sum_{i=0}^v \frac{f^{(i)}(x)}{i!} (t-x)^i + \frac{f^{(v)}(\xi) - f^{(v)}(x)}{v!} (t-x)^v.$$

For $t \in [0, \infty) \setminus (a - \eta, b + \eta)$ and $x \in [a, b]$, we define

$$h(t, x) = f(t) - \sum_{i=0}^v \frac{f^{(i)}(x)}{i!} (t-x)^i.$$

Now,

$$\begin{aligned} \tilde{S}_{n,p}^{(r)}(f(t); x) - f^{(r)}(x) &= \left(\sum_{i=0}^v \frac{f^{(i)}(x)}{i!} \tilde{S}_{n,p}^{(r)}((t-x)^i; x) - f^{(r)}(x) \right) \\ &\quad + \tilde{S}_{n,p}^{(r)} \left(\frac{f^{(v)}(\xi) - f^{(v)}(x)}{v!} (t-x)^v \chi(t); x \right) \\ &\quad + \tilde{S}_{n,p}^{(r)}(h(t, x)(1 - \chi(t)); x) \\ &:= \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned}$$

$$\begin{aligned} |\Sigma_2| &\leq \tilde{S}_{n,p}^{(r)} \left(\frac{|f^{(v)}(\xi) - f^{(v)}(x)|}{v!} |t-x|^v \chi(t); x \right) \\ &\leq \frac{\omega_{f^{(v)}}(\delta; (a - \eta, b + \eta))}{v!} \tilde{S}_{n,p}^{(r)} \left(\left(1 + \frac{|t-x|}{\delta} \right) |t-x|^v; x \right) \\ &\leq \frac{\omega_{f^{(v)}}(\delta; (a - \eta, b + \eta))}{v!} \end{aligned}$$

$$\begin{aligned} &\left[\frac{n}{G_x} \sum_{k=0}^{\infty} |q_{n,pk}^{(r)}(x)| \int_0^{\infty} q_{n,pk}(t) (|t-x|^v + \delta^{-1} |t-x|^{v+1}) dt \right. \\ &\quad \left. + e^{-nx} (x^v + \delta^{-1} x^{v+1}) \right], \delta > 0. \end{aligned}$$

Now, for $s = 0, 1, 2, \dots$, we have:

$$\begin{aligned} & \frac{n}{G_x} \sum_{k=1}^{\infty} q_{n,pk}(x) |pk \\ & - nx |^j \int_0^{\infty} q_{n,pk}(t) |t \\ & - x|^s dt \\ & \leq \left(\frac{1}{G_x} \sum_{k=0}^{\infty} q_{n,pk}(x) (pk - nx)^{2j} \right)^{\frac{1}{2}} \\ & \times \left(\frac{n}{G_x} \sum_{k=0}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk}(t) (t \\ & - x)^{2s} dt \right)^{\frac{1}{2}} \\ & = O \left(n^{\frac{j}{2}} \right) O \left(n^{-\frac{s}{2}} \right) \\ & = O \left(n^{\frac{(j-s)}{2}} \right) \text{ uniformly on } [a, b]. \end{aligned}$$

Therefore, by using **Lemma 3**, (6) and substitute k by pk , we get:

$$\frac{n}{G_x} \sum_{k=0}^{\infty} |q_{n,pk}^{(r)}(x)| \int_0^{\infty} q_{n,pk}(t) |t - x|^s dt$$

Choosing $\delta = n^{-\frac{1}{2}}$ and applying (7), we are led to

$$\begin{aligned} & \|\Sigma_2\|_{C[a,b]} \\ & \leq \frac{\omega_{f(v)} \left(n^{-\frac{1}{2}}; (a - \eta, b + \eta) \right)}{v!} \left[O \left(n^{\frac{(r-v)}{2}} \right) \right. \\ & \left. + n^{\frac{1}{2}} O \left(n^{\frac{(r-v-1)}{2}} \right) + O(n^{-m}) \right], \end{aligned}$$

for any $m > 0$,

$$\leq C_2 n^{-\frac{(r-v)}{2}} \omega_{f(v)} \left(n^{-\frac{1}{2}}; (a - \eta, b + \eta) \right).$$

$$\begin{aligned} & \leq \frac{n}{G_x} \sum_{k=0}^{\infty} \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i |pk \\ & - nx |^j \frac{|Q_{i,j,r}(x)|}{x^r} q_{n,pk}(x) \int_0^{\infty} q_{n,pk}(t) (t \\ & - x)^s dt \\ & \leq \left(\sup_{\substack{2i+j \leq r \\ i,j \geq 0}} \sup_{x \in [a,b]} \frac{|Q_{i,j,r}(x)|}{x^r} \right) \left(\sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i \left(\frac{n}{G_x} \sum_{k=0}^{\infty} q_{n,pk}(x) |pk \right. \right. \\ & \left. \left. - nx |^j \int_0^{\infty} q_{n,pk}(t) |t - x|^s dt \right) \right) \\ & = C \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i O \left(n^{\frac{(j-s)}{2}} \right) \\ & = O \left(n^{\frac{(r-s)}{2}} \right), \text{ uniformly on } [a, b], \end{aligned} \tag{7}$$

since

$$\begin{aligned} & \sup_{\substack{2i+j \leq r \\ i,j \geq 0}} \sup_{x \in [a,b]} \frac{|Q_{i,j,r}(x)|}{x^r} \\ & = M(x), \text{ but fixed.} \end{aligned}$$

Since $t \in [0, \infty) \setminus (a - \eta, b + \eta)$, we can

choose $\delta > 0$ in such a way that

$$|t - x| \geq \delta \text{ for all } x \in [a, b].$$

Thus,

$$\begin{aligned} & |\Sigma_3| \\ & \leq \frac{n}{G_x} \sum_{k=0}^{\infty} \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i |pk \\ & - nx |^j \frac{|Q_{i,j,r}(x)|}{x^r} q_{n,pk}(x) \int_{|t-x| \geq \delta} q_{n,pk}(t) |h(t, x)| dt \\ & + e^{-nx} |h(0, x)|. \end{aligned}$$

For $|t - x| \leq \delta$, we can find a constant $C > 0$ such that $|h(t, x)| \leq C e^{ht}$.

Finally using Schwarz inequality for integration and then for summation , we

get:

$$|\Sigma_3| = O(n^{-s}), s > 0 \text{ uniformly on } [a, b].$$

Combining the estimates of $\Sigma_1, \Sigma_2, \Sigma_3$, the required result is immediate.

■

References:

- [1] H. S. Kasana, G. Prasad, P. N. Agrawal and A. Sahai, *On modified Szász operators*, in: (Proceedings of the International Conference on Mathematical Analysis and its Applications) Kuwait Eds. S. M. Mazhar, A. Hamoui and N. S.

Faour; Pergamon Press, Oxford, (1985), 29-41.

[2] H. A. Naser. " On Generalization for Some Szász Type Operators " Ms. c. thesis , University of Basrah, 2011.

[3] Rempulska and Graczyk, Approximation by modified Szász Mirakyan operator, J. Inequal. Pure and Appl. Math., 10(3)(2009), Art. 61, 1-8.

[4] Szász, Generalization of S. Bernstein's polynomials to the infinite interval, J. Res. Nat. Bur. Standard, 45(1950), 239-245.

التقريب المتعدد باستخدام تعميم لمؤثر من النمط زاز

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الخلاصة

في هذا البحث، نقدم تعميماً للمؤثر من النمط مجموع-تكامل Szász ونرمز له $\tilde{S}_p(f(t); x)$. أولاً، ندرس تقارب المؤثر وبعد ذلك نجد صيغة تكرارية لهذا المؤثر وفي النهاية نثبت صيغة فرونوشكي للتقارب (Voronovskaja-type asymptotic formula) ونحصل على الخطأ المخمن في حدود مقياس الاستمرارية للدوال المقربة.