Applications of the Operator $r\Phi_s$ in $q$-identities

Husam L. Saad*, Hassan J. Hassan
Department of Mathematics, College of Science, Basrah University, Basrah, Iraq
* Corresponding Author Husam L. Saad, E-mail: hus6274@hotmail.com

Doi:10.29072/basjs.202112

Abstract

In this paper, we set up the general operator $r\Phi_s$, and then we find some of its operator identities that will be used to generalize some well-known $q$-identities, such as Cauchy identity, Heine’s transformation formula and the $q$-Pfaff-Saalschütz summation formula. By giving special values to the parameters in the obtained identities, some new results are achieved and/or others are recovered.
1. Introduction

We adopt the following notations and terminology in [8]. We assume that \(0 < q < 1\). The \(q\)-shifted factorial is given by

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k).
\]

and the multiple \(q\)-shifted factorials is given by

\[
(a_1, a_2, \ldots, a_r; q)_m = (a_1; q)_m(a_2; q)_m \cdots (a_r; q)_m.
\]

where \(m \in \mathbb{Z}\) or \(\infty\).

The basic hypergeometric series \(r\phi_s\) is defined as follows [8]:

\[
r\phi_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \\ q, x \end{array} \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left[ \frac{(-1)^k q^{k(k-1)}}{2} \right]^{1+s-r} x^k,
\]

where \(r, s \in \mathbb{N}\); \(a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{C}\); and none of the denominator factors evaluate to zero. The above series is absolutely convergent for all \(x \in \mathbb{C}\) if \(r < s + 1\), for \(|x| < 1\) if \(r = s + 1\) and for \(x = 0\) if \(r > s + 1\).

The \(q\)-binomial coefficient is presented as follows [8]:

\[
[n]_k = \begin{cases} 
\frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}, & \text{if } 0 \leq k \leq n; \\
0, & \text{otherwise},
\end{cases}
\]

where \(n, k\) are nonnegative integers.

In this paper, we will repeatedly use the following equations [8]:

\[
(b; q)_{-k} = \frac{(-1)^k q^{k(k-1)/2} (q/b)^k}{(q/b; q)_k}.
\]

\[
(b; q)_{n-k} = \frac{(b; q)_n}{(q^{1-n}/b; q)_k} (-1)^k q^{k(n-k)/2} \left( \frac{q}{b} \right)^k.
\]

\[
(q^{-n}; q)_k = \frac{(q; q)_n}{(q; q)_{n-k}} (-1)^k q^{k(n-k)/2}.
\]

\[
(bq^{-n}; q)_{\infty} = (-1)^n b^n q^{(n+1)/2} (q/b; q)_n (b; q)_{\infty}.
\]

The Cauchy identity is given by:

\[
\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1.
\]
The special case of the Cauchy identity (1.5), given by Euler, is [8]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n)}_q}{(q;q)_n} x^n = (x; q)_\infty.
\]  

(1.6)

The q-Chu-Vandermonde’s identities are [8]

\[
2\phi_1\left(\frac{q^{-n}, b}{c, c q^n/b}; q, c q^n/b\right) = \frac{(c/b; q)_n}{(c; q)_n}, \quad |c/b| < 1.
\]  

(1.7)

\[
2\phi_1\left(\frac{q^{-n}, b}{c, q}; q, q\right) = \frac{(c/b; q)_n}{(c; q)_n} b^n.
\]  

(1.8)

The q-Pfaff-Saalschütz sum is given by [8]

\[
3\phi_2\left(\frac{q^{-n}, a, b}{c, q^{1-n} a b/c}; q, q\right) = \frac{(c/a, c/b; q)_n}{(c, c/a b; q)_n}.
\]  

(1.9)

The q-Gauss summation formula is given by [8]

\[
2\phi_1\left(\frac{a, b}{c, q/c a b}; q, q/c a b\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/a b; q)_\infty}, \quad \left|\frac{c}{a b}\right| < 1.
\]  

(1.10)

Heine’s transformation formula is given by [8]

\[
2\phi_1\left(\frac{a, b}{c, z}; q, z\right) = \frac{(c/b, z b; q)_\infty}{(c, z; q)_\infty} 2\phi_1\left(\frac{a b z/c, b}{z b, q, c/b} ; q, c/b\right).
\]  

(1.11)

where \(\max\{\left|a\right|, \left|c/b\right|\} < 1\).

The transformation formula [8, Appendix III, equation (III.9)] is given by:

\[
3\phi_2\left(\frac{a, b, c}{d, e}; q, d e/abc\right) = \frac{(e/a, d e/b c; q)_\infty}{(e, d e/abc; q)_\infty} 3\phi_2\left(\frac{a, d/b, d/c}{d, d e/b c; q}_\infty\right).
\]  

(1.12)

**Definition 1.1** ([2], [3], [10]). The \(D_q\) operator or the q-derivative is defined as follows:

\[
D_q[f(a)] = \frac{f(a) - f(aq)}{a}.
\]  

(1.13)

**Theorem 1.2** ([2], [10]). For \(n \geq 0\), we have

\[
D_q^n[f(a) g(a)] = \sum_{k=0}^{n} \binom{n}{k} q^{k(n-k)} D_q^k[f(a)] D_q^{n-k}[g(a q^k)].
\]  

(1.14)

**Theorem 1.3** ([2], [16]). Let \(D_q\) be defined as in (1.13), then
In 2010, Fang [5] defined the finite operator as follows:

**Definition 1.4** [5]. The \( q \)-exponential operator \( _1 \Phi_0 \left( q^{-M} \right) \) is defined by:

\[
_1 \Phi_0 \left( q^{-M} \right) = \sum_{k=0}^{M} \frac{(q^{-M}; q)_k}{(q; q)_k} (cD_q)^k.
\]  

Fang used the \( q \)-exponential operator \( _1 \Phi_0 \left( q^{-M} \right) \) to prove the following result:

**Theorem 1.5** [5]. Let \( _1 \Phi_0 \left( q^{-M} \right) \) be defined as in (1.16), then

\[
_3 \Phi_2 \left( \frac{q^{-M}, c_1}{d_2}, xd_1; q, cD_2 \right) = \frac{(cD_2, q)_M}{(cD_1, q)_M} \frac{d_2}{d_1}^M \frac{cD_1, q^{-M}, xc_1; q, cD_1}{cD_2 q^{-M}, xc_1; q, cD_2}.
\]

In 2010, Zhang and Yang [15] constructed the finite \( q \)-Exponential Operator \( 2E_1 \left[ q^{-N} ; W ; q, cD_q \right] \) with two parameters as follows:

**Definition 1.6** [15]. The finite \( q \)-Exponential Operator \( 2E_1 \left[ q^{-N} ; W ; q, cD_q \right] \) is defined by

\[
2E_1 \left[ q^{-N} ; W ; q, cD_q \right] = \sum_{n=0}^{N} \frac{(q^{-N}, W; q)_n}{(q, v; q)_n} (cD_q)^n.
\]  

Zhang and Yang used the operator \( 2E_1 \left[ q^{-N} ; W ; q, cD_q \right] \) to get a generalization of \( q \)-Chu-Vandermond formula (1.8) as follows:

**Theorem 1.7** [15]. Let \( 2E_1 \left[ q^{-N} ; W ; q, cD_q \right] \) be defined as in (1.18), then

\[
\sum_{n=0}^{N} \sum_{k=0}^{n} (q^{-n}, a; q)_m (q^{-N}, W; q)_k \frac{c^k q^{m+mk}}{(q, c; q)_m (q, v; q)_k} = a^n W^n \left( \frac{c/a; q}_n \frac{v/w; q}_N \right) \frac{4 \Phi_2 \left( \frac{q^{-N}, q^{1-n}, aq}{c, c} \frac{aq^{1-n}}{w^{1-n}} ; q, \frac{c}{v} \right)}{c, c, w, v}.
\]
Also, by using the operator $\mathcal{E}_1 \left[ q^{-N} \frac{v^{-1}}{v}; q, c D_q \right]$, they obtained the following result:

$$\phi_1 \left( \frac{q^{-N} w}{v}; q, c \right) = w^N \frac{(v/w; q)_N}{(v; q)_N} \phi_1 \left( \frac{q^{-N} w}{c}; q, \frac{c}{v} \right)$$

(1.20)

In 2016, Li-Tan [9] constructed the generalized $q$-exponential operator $T \left[ \frac{u,v}{w} | q; c D_q \right]$ with three parameters as follows:

**Definition 1.8** [9]. The generalized $q$-exponential operator $T \left[ \frac{u,v}{w} | q; c D_q \right]$ is defined by

$$T \left[ \frac{u,v}{w} | q; c D_q \right] = \sum_{n=0}^{\infty} \frac{(u,v; q)_n}{(q,w; q)_n} (c D_q)^n.$$  

(1.21)

Li and Tan used the generalized $q$-exponential operator $T \left[ \frac{u,v}{w} | q; c D_q \right]$ to get a generalization for $q$-Chu-Vandermonde sum (1.8), as follows:

**Theorem 1.9** [9]. Let $T \left[ \frac{u,v}{w} | q; c D_q \right]$ be defined as in (1.21), then

$$\sum_{k=0}^{n} \frac{(q^{-n}, x; q)_k}{(q,c; q)_k} q^k 2 \phi_1 \left[ \frac{u,v}{w} ; q, t q^k \right] = x^n \frac{(c/x; q)_n}{(c; q)_n} \sum_{i,k \geq 0} \frac{(u,v; q)_{i+k}}{(q; q)_{i+k}} \frac{(q^{-1}/c, qx/c; q)_k}{(q, q^{-1}x/c; q)_k} t^{i+k} \left( \frac{q}{c} \right)^i.$$  

(1.22)

The Cauchy polynomials $P_n(x, y)$ is defined by [7]

$$P_n(x, y) = \begin{cases} (x-y)(x-qy)(x-q^2y) \cdots (x-q^{n-1}y), & \text{if } n > 0; \\ 1, & \text{if } n = 0. \end{cases}$$

(1.23)

In 1983, Goulden and Jackson [7] gave the following identity:

$$P_n(x, y) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] (-1)^k q^k y^k x^{n-k}.$$  

The generating function for Cauchy polynomials $P_n(x, y)$ [1] is

$$\sum_{k=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(xt; q)_\infty}, \quad |xt| < 1.$$  

(1.24)

In 2003, Chen et al [1] introduced the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ as:

$$h_n(x,y|q) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] P_k(x,y),$$

where $P_k(x,y)$ is defined as in (1.23). In 2010, Saad and Sukhi [11] gave another formula for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ as:

$$h_n(x,y|q) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] (y; q)_k x^{n-k}.$$  

The generating function for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ is [1]
The generalized Al-Salam–Carlitz $q$-polynomials $\phi_n^{(a,b)}(x,y)$ was introduced in 2020 by Srivastava and Arjika [14] as
\[
\phi_n^{(a,b)}(x,y) = \sum_{k=0}^{n} \frac{[n]_k!}{[k]_k!} \frac{(a_1, a_2, \cdots, a_{s+1}; q)_k}{(b_1, b_2, \cdots, b_s; q)_k} x^k y^{n-k},
\]
which has the following generating function:
\[
\sum_{n=0}^{\infty} \phi_n^{(a,b)}(x,y) \frac{t^n}{(q; q)_n} = \frac{1}{(yt; q)_\infty} \sum_{s+1} \Phi_s \left( \begin{array}{c} a_1, a_2, \cdots, a_{s+1} \\ b_1, b_2, \cdots, b_s \end{array}; q, xt \right),
\]
where $\max[|xt|, |yt|] < 1$.

The paper is organized as follows. In section 2, we built the general operator $\Phi_s \left( \begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, cD_q \right)$. We also provide some operator identities, which will be used in section 3. In section 3, we generalize some well-known $q$-identities, such as Cauchy identity, Heine’s transformation formula and the $q$-Pfaff-Saalschütz summation formula. Then, in these generalizations, we may assign the parameters unique values, we get several results.

2. The General Operator $\Phi_s$ and its Identities

In this section, we establish the general operator $\Phi_s \left( \begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, cD_q \right)$. We also give some identities to this operator, which will be used in the next section.

**Definition 2.1** We define the generalized $q$-operator $\Phi_s$ as follows:
\[
\Phi_s \left( \begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, cD_q \right) = \sum_{n=0}^{\infty} \frac{W_n}{(q; q)_n} \left( -1 \right)^n q \left( \frac{n}{2} \right) \left( cD_q \right)^n,
\]
where $W_n = \frac{(a_1, \cdots, a_r; q)_n}{(b_1, \cdots, b_s; q)_n}$.

Some special values may be given to the general $q$-operator $\Phi_s$ to obtain several previously specified operators, as follows:

- Setting $r = 1$, $s = 0$, $a_1 = 0$ and $c = b$, we get on the exponential operator $T(bD_q)$ defined by Chen and Liu [2] in 1997.
- If $r = 1, s = 0$ and $a_1 = b$, we get on the Cauchy operator $\Phi_0 \left( b; q, cD_q \right)$ which was defined by Fang[4] in 2008.
• If \( r = 1, s = 0 \) and \( a_1 = q^{-M} \), we get on the finite operator 
\( 1 \Phi_0 \left( q^{-M} ; q, cD_q \right) \)

• If \( r = 2, s = 1 \), \( a_1 = q^{-N} \), \( a_2 = w \) and \( b_1 = v \), we get on the
finite exponential operator 
\( 2 \mathcal{E}_1 \left[ q^{-N}, w; q, cD_q \right] \) with two parameters

• If \( r = s = 0 \), we get on the \( q \)-exponential operator \( R(bDq) \) which is defined by

• Setting \( r = s + 1 \), we get the generalized \( q \)-operator \( F(a_0, \ldots, a_s; b_1, \ldots, b_s; cD_q) \)
described by Fang [6] in 2014 and the homogeneous \( q \)-difference
operator \( \mathbb{T}(a, b, cD_q) \) specified by Srivastava and Arjika [14] in 2020.

• If \( r = 2, s = 1 \), \( a_1 = u, a_2 = v \) and \( b_1 = w \), we get on the
generalized exponential operator 
\( \mathcal{T} \left[ u, v; q, cD_q \right] \) with three parameters

• Setting \( r = 3, s = 2 \), \( a_1 = a, a_2 = b, a_3 = c \), \( b_1 = d, b_2 = e \) and \( c = f \), we get the
operator \( \phi \left( a, b, c, d, e, cD_q \right) \) with five parameters

The following operator identities will be derived using \( q \)-Leibniz formula (1.14):

**Theorem 2.2** Let 
\( r \Phi_s \left( a_1, \ldots, a_r; b_1, \ldots, b_s ; q, cD_q \right) \) be defined as in (2.1), then

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{W_{n+k}}{(q;q)_n} \left( \frac{v/t, aw; q}{q, av; q} \right)_k \left( \frac{u/w, q}{q, au; q} \right)_n \left( -1 \right)^{n+k} q^{\frac{n+k}{2}} \right]^{1+s-r} (cw)^n (ct)^k
\]

provided that \( \max(\left| at \right|, \left| aw \right|) < 1 \).

**Proof.**

\[
\sum_{n=0}^{\infty} \frac{W_n}{(q;q)_n} \left( -1 \right)^n q^{\frac{n}{2}} c^n q^n (ct)^k
\]

(by using (2.1))
Setting in equation (2.2), we get the following corollary:

**Corollary 2.2.1** Let $r\Phi_s\left(\begin{array}{c}a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array}; q, cD_q\right)$ be defined as in (2.1), then

$$r\Phi_s\left(\begin{array}{c}a_1,\ldots,a_r \\ b_1,\ldots,b_s ; q, cD_q\end{array}\right)\left(\frac{(av;q)_\infty}{(at;q)_\infty}\right) = \frac{(av;q)_\infty}{(at;q)_\infty}\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{W_{n+k}}{(q;q)_n} \left(\frac{(v/t, aw;q)_k}{(at, aw; q)_k}\right) \left(\frac{(u/w; q)_n}{(au; q_n)^{n+k}}\right)\left[(-1)^{n+k}q^{\frac{n+k}{2}}\right]^{1+s-r} (cw)^n (ct)^k,$$

where $\max\{|at|, |aw|\} < 1$.

In view of symmetry of $t$ and $w$ on the left hand side of equation (2.4), we get the following formula:

**Theorem 2.3**

$$\sum_{n,k \geq 0} \frac{W_{n+k}}{(q;q)_n} \left[(-1)^{n+k}q^{\frac{n+k}{2}}\right]^{1+s-r} \frac{(v/t, aw; q)_k}{(q, aw; q)_k} (cw)^n (ct)^k$$

$$= \sum_{n,k \geq 0} \frac{W_{n+k}}{(q;q)_n} \left[(-1)^{n+k}q^{\frac{n+k}{2}}\right]^{1+s-r} \frac{(v/w, at; q)_k}{(q, av; q)_k} (ct)^n (cw)^k. \quad (2.5)$$

- If $r = 1, s = 0$ in equation (2.5) and then using (1.5), we get Hall’s transformation (1.12).
- If $r = 1, s = 0$ and $a_1 = q^{-N}$ in equation (2.5), then using equations (1.4) and (1.5), we get Theorem 3.5. obtained by Fang [5] (equation (1.17)).

**Theorem 2.4** Let $r\Phi_s\left(\begin{array}{c}a_1,\ldots,a_r \\ b_1,\ldots,b_s ; q, cD_q\end{array}\right)$ be defined as in (2.1), then

$$r\Phi_s\left(\begin{array}{c}a_1,\ldots,a_r \\ b_1,\ldots,b_s ; q, cD_q\end{array}\right)\left(\frac{a^n (ax;q)_\infty}{(ay;q)_\infty}\right) = a^n \left(\frac{(ax;q)_\infty}{(ay;q)_\infty}\right)$$
Proof.

\[
\begin{align*}
\Phi_s \left( a_1, \ldots, a_r, b_1, \ldots, b_s ; q, cD_q \right) \\
= \sum_{i=0}^{\infty} \frac{W_i}{(q; q)_i} \left[ (-1)^i q^{i/2} \right]^{1+s-r} c^i D_q \left\{ a^n \frac{(ax; q)^{\infty}}{(ay; q)^{\infty}} \right\} \\
= \sum_{i=0}^{\infty} \frac{W_i}{(q; q)_i} \times \left[ (-1)^i q^{i/2} \right]^{1+s-r} c^i \\
\times \sum_{j=0}^{\infty} q^{j^2-ij} \left\{ \frac{(axq^j; q)^{\infty}}{(ayq^j; q)^{\infty}} \right\} \\
= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{W_i}{(q; q)_i} \left[ (-1)^i q^{i/2} \right]^{1+s-r} q^{j^2-ij} \left\{ \frac{(axq^j; q)^{\infty}}{(ayq^j; q)^{\infty}} \right\} \\
= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{W_{i+j}}{(q; q)_i} \left[ (-1)^i q^{i/2} \right]^{1+s-r} q^{-ij} \left\{ \frac{(axq^{i+j}; q)^{\infty}}{(ayq^j; q)^{\infty}} \right\} \\
= a^n \frac{(ax; q)^{\infty}}{(ay; q)^{\infty}} \sum_{i,j=0}^{\infty} \frac{W_{i+j}}{(q; q)_i} \left[ (-1)^i q^{i+j} \left( \frac{axq^{i+j}}{ayq^j} \right) \right]^{1+s-r} \left\{ \frac{(x/y; q)_i (axq^{i+j}; q)^{\infty}}{(ayq^j; q)^{\infty}} \right\} \\
= a^n \frac{(ax; q)^{\infty}}{(ay; q)^{\infty}} \sum_{i,j=0}^{\infty} \frac{W_{i+j}}{(q; q)_i} \left[ (-1)^i j q^{i+j} \left( \frac{axq^{i+j}}{ayq^j} \right) \right]^{1+s-r} \left\{ \frac{c}{a} \right\}^j, |ay| < 1. \quad (2.6)
\end{align*}
\]

Setting \( x = 0 \) in equation (2.6), we get the following corollary:

**Corollary 2** Let \( \Phi_s \left( a_1, \ldots, a_r, b_1, \ldots, b_s ; q, cD_q \right) \) be defined as in (2.1), then

\[
\begin{align*}
\Phi_s \left( a_1, \ldots, a_r, b_1, \ldots, b_s ; q, cD_q \right) \\
= a^n \frac{(ax; q)^{\infty}}{(ay; q)^{\infty}} \sum_{i,j=0}^{\infty} \frac{W_{i+j}}{(q; q)_i} \left[ (-1)^i q^{i+j} \left( \frac{axq^{i+j}}{ayq^j} \right) \right]^{1+s-r} \left\{ \frac{c}{a} \right\}^j, |ay| < 1. \quad (2.7)
\end{align*}
\]

### 3. Applications in q-Identities

In this section, we aim to generalize some well-known q-identities such as Cauchy identity.
Heine’s transformation of \( {}_2\phi_1 \) series and \( q \)-Pfaff-Saalschütz sum by using the general operator \( {}_r\Phi_s \). Then, some special results are obtained from these generalizations, some new ones and others are known.

### 3.1 Generalization of Cauchy Identity

**Theorem 3.1** (Generalization of Cauchy identity). Let Cauchy identity be defined as in (1.5), then

\[
\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} x^k \sum_{i,j \geq 0} \frac{W_{i+j} (b/c; q)_i (xb; q)_{i+j}}{(q; q)_i (xb; q)_{i+j}} \left[ (-1)^{i+j} q^{i+j/2} \right]^{1+s-r} (xc; q)_j \left[ k \right] (dc)^i \left( \frac{d}{x} \right)^j \\
= \frac{(xa; q)_\infty}{(x; q)_\infty} \sum_{i,j \geq 0} \frac{W_{i+j} (b/c; q)_i (xb; q)_{i+j}}{(q; q)_i (xb; q)_{i+j}} \left[ (-1)^{i+j} q^{i+j/2} \right]^{1+s-r} \frac{(a, xc; q)_j}{(q, xa; q)_j} (dc)^i d^j.
\] (3.1)

**Proof.** Multiply Cauchy identity by \( \frac{(xb; q)_\infty}{(xc; q)_\infty} \).

\[
\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} x^k \frac{(xb; q)_\infty}{(xc; q)_\infty} = \frac{(ax, xb; q)_\infty}{(x, xc; q)_\infty}.
\] (3.2)

Applying the operator \( {}_r\Phi_s \left( b_1, \ldots, b_s; q, dD_q \right) \) on both sides of (3.2), we get

\[
\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} {}_r\Phi_s \left( \frac{a_1, \ldots, a_r}{b_1, \ldots, b_s; q, dD_q} \right) \left\{ x^k \frac{(xb; q)_\infty}{(xc; q)_\infty} \right\} \\
= {}_r\Phi_s \left( \frac{a_1, \ldots, a_r}{b_1, \ldots, b_s; q, dD_q} \right) \left\{ \frac{(ax, xb; q)_\infty}{(x, xc; q)_\infty} \right\}.
\] (3.3)

By using (2.4), we get

\[
\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} {}_r\Phi_s \left( \frac{a_1, \ldots, a_r}{b_1, \ldots, b_s; q, dD_q} \right) \left\{ x^k \frac{(xb; q)_\infty}{(xc; q)_\infty} \right\} \\
= x^k \frac{(xb; q)_\infty}{(xc; q)_\infty} \sum_{i,j \geq 0} \frac{W_{i+j} (b/c; q)_i (xb; q)_{i+j}}{(q; q)_i (xb; q)_{i+j}} \left[ (-1)^{i+j} q^{i+j/2} \right]^{1+s-r} (xc; q)_j \left[ k \right] (dc)^i \left( \frac{d}{x} \right)^j
\] (3.4)

and using (2.2), we get

\[
{}_r\Phi_s \left( \frac{a_1, \ldots, a_r}{b_1, \ldots, b_s; q, dD_q} \right) \left\{ \frac{(ax, xb; q)_\infty}{(x, xc; q)_\infty} \right\} \\
= \frac{(xa; q)_\infty}{(x; q)_\infty} \sum_{i,j \geq 0} \frac{W_{i+j} (b/c; q)_i (xb; q)_{i+j}}{(q; q)_i (xb; q)_{i+j}} \left[ (-1)^{i+j} q^{i+j/2} \right]^{1+s-r} \frac{(a, xc; q)_j}{(q, xa; q)_j} (dc)^i d^j.
\] (3.5)

Substituting (3.4) and (3.5) into (3.3) the proof completed.

• If \( d = 0 \) in equation (3.1), we obtain Cauchy identity.
• If \( b = 0 \) and then \( c = 0 \) in equation (3.1), we obtain the following formula:

**Corollary 3.1.3**

\[
\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} \sum_{j=0}^{\infty} W_j \binom{k}{j} (-1)^j q^{j(\frac{1}{2})} d^j x^{k-j} = \frac{(xa;q)_\infty}{(x;q)_\infty} \sum_{j=0}^{\infty} \frac{W_j \, (a;q)_j}{(q;q)_j (xa;q)_j} (-1)^j q^{j(\frac{1}{2})} d^j .
\] (3.6)

• If \( r = s = 0, a = 0, x \to xt \) and \( d \to yt \) in equation (3.6), we get the generating function for Cauchy polynomials \( P_k(x,y) \) (1.26).

• If \( r = 1, s = 0 \) and \( a = 0 \) then replacing \( x, a_1, d \) by \( xt, y, t \) respectively, in equation (3.6), we get on the generating function for bivariate Rogers-Szeg\text{"{o}} polynomials \( h_k(x,y|q) \) (1.25).

• If \( r = s + 1, a = 0, x \to yt \) and then \( d \to xt \) in equation (3.6), we get the generating function for the generalized Al-Salam–Carlitz \( q \)-polynomials \( \phi^{(a,b)}_n(x,y) \) (1.26).

### 3.2 Generalization of Heine’s Transformation of \( \phi_1 \) Series

**Theorem 3.2** (Generalization of Heine’s transformation of \( \phi_1 \) series). Let Heine’s identity be defined as in (1.11), then

\[
\sum_{k=0}^{\infty} \frac{(a,b;q)_k}{(q,c;q)_k} z^k \sum_{n,l=0}^{\infty} \frac{W_{n+l} \, (zq^k)}{(a;b)_n \, (c;q)_n} \binom{k}{i} (zq^k)^i \left( \frac{-1}{2} \right)^{n+i} q^{\frac{n(i+1)}{2}} + \frac{(dbq)^n}{(d/z)^i}.
\] (3.7)

**Proof.** Rewrite Heine’s formula as follows.

\[
\sum_{k=0}^{\infty} \frac{(a,b;q)_k}{(q,c;q)_k} z^k = \frac{(c/b;q)_\infty}{(zq^k;q)_\infty} \sum_{k=0}^{\infty} \frac{(b;q)_k}{(q;q)_k} (c/b)^k \frac{(abz/c;q)_\infty}{(zq^k;q)_\infty}.
\] (3.8)

Applying the general operator \( \phi_s \left( \begin{array}{cc} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} ; q, DD_q \right) \) to both sides of the equation (3.8) gives:

\[
\sum_{k=0}^{\infty} \frac{(a,b;q)_k}{(q,c;q)_k} \phi_s \left( \begin{array}{cc} a_1, \cdots, a_r \\ b_1, \cdots, b_s ; q, DD_q \end{array} \right) \left( \frac{z^k}{(zq^k;q)_\infty} \right).
\]
Using (2.7), we get
\[
\sum_{k=0}^{\infty} (\frac{b; q)_k}{(q; q)_k} (\frac{c/b; q)_k}{(z/bq^k; q)_k} \Phi_1 \left( \frac{a_1, \ldots, a_r}{b_1, \ldots, b_s; q, dD_q} \middle| \frac{abz/c; q)_k}{(zbq^k, z; q)_k} \right).
\] (3.9)

and using (2.4), we get
\[
\sum_{k=0}^{\infty} \frac{z^k}{(zbq^k, z; q)_k} \sum_{n, l \geq 0} W_{n+l} \left( \frac{k!}{q; q)_n \left( zbq^k; q \right)_l \left( -1 \right)^{n+l} q^{\frac{n+l}{2}} \right)^{1+s-r} \left( dbq^k \right)^n d^n \left( \frac{dab/c}{q^k} \right)^i. \] (3.10)

Substituting (3.10) and (3.11) in equation (3.9) the proof is completed.

- If \( r = s + 1, a = 0, z \rightarrow yt, d \rightarrow xt, c \rightarrow cb \) and then \( b = 0 \) in equation (3.7), we get the generating function for the generalized Al-Salam–Carlitz q-polynomials \( \phi_n^{(a,b)}(x,y) \) (1.26).

- If \( r = 1, s = 0 \) in equation (3.7), we get the following identity:

**Corollary 3.2.4**
\[
\sum_{k=0}^{\infty} \frac{(a,b, db; q)_k}{(q,c, a_1 db; q)_k} z^k \frac{3\Phi_1}{(a_1 dbq^k; q)_k} \left( q^{-k}, a_1, zbq^k ; q, dqb^k / z \right) = \frac{(a_1 d, db, c/b, zb; q)_k}{(d, a_1 db, c, z; q)_k} \sum_{k=0}^{\infty} \frac{(abz/c, q)_k}{q, zb; q)_k} (c/b)_k^k \frac{3\Phi_2}{(q^{-k}, a_1, z; abz/c, a_1 d; q, dqb^k / c)}.
\]

### 3.3 Generalization of q-Pfaff-Saalschütz Sum

**Theorem 3.3** (Generalization of q-Pfaff-Saalschütz sum). Let q-Pfaff-Saalschütz sum be defined as in (1.9), then
\[
\sum_{k=0}^{\infty} \frac{(q^n, a, b; q)_k}{(q, c, abq^{-n}/c; q)_k} q^k \sum_{i,j \geq 0} W_{i+j} \left( \frac{(q^{-n+k}, q)_i}{(abq^{-n+k}/c; q)_i} \right) \left( yq^{-k}, abq/c; q \right)_j \left( -1 \right)^{i+j} q^{i+j} \left( dbq/c \right)^i (dq^k)^j
\]
Proof. Multiplaying \( q \)-Saalschütz identity (1.9) by \((ay; q)_\infty\), we have

\[
\sum_{k=0}^{\infty} \left( \begin{array}{c}
q^{-n}k \\
q^n, b; q \end{array} \right)_k q^k \frac{(ay, abq^{1-n+k}/c; q)_\infty}{(aq^k, abq/c; q)_\infty} = \frac{b^n(c/b; q)_n (aq^{1-n}/c, ay; q)_\infty}{(c; q)_n (a, aq/c; q)_\infty}.
\]

Applying the general operator \( \Phi_S \left( \begin{array}{c}
a_1, \ldots, a_r \\
b_1, \ldots, b_s; q, dD_q \end{array} \right) \) to both sides of equation (3.13) gives:

\[
\sum_{k=0}^{n} \left( \begin{array}{c}
q^{-n}k \\
q^n, c; q \end{array} \right)_k q^k \Phi_S \left( \begin{array}{c}
a_1, \ldots, a_r \\
b_1, \ldots, b_s; q, dD_q \end{array} \right) \left( \begin{array}{c}
(ay, abq^{1-n+k}/c; q)_\infty \\
(aq^k, abq/c; q)_\infty \end{array} \right) = \frac{(-c)^n q^n(a)_n}{(c; q)_n} \Phi_S \left( \begin{array}{c}
a_1, \ldots, a_r \\
b_1, \ldots, b_s; q, dD_q \end{array} \right) \left( \begin{array}{c}
(aq^{1-n}/c, ay; q)_\infty \\
(a, aq/c; q)_\infty \end{array} \right).
\]

Using (2.2), we get

\[
\frac{(ay, abq^{1-n+k}/c; q)_\infty}{(aq^k, abq/c; q)_\infty} \sum_{k=0}^{n} \left( \begin{array}{c}
q^{-n}k \\
q^n, c; q \end{array} \right)_k q^k \Phi_S \left( \begin{array}{c}
a_1, \ldots, a_r \\
b_1, \ldots, b_s; q, dD_q \end{array} \right) \left( \begin{array}{c}
(ay, abq^{1-n+k}/c; q)_\infty \\
(aq^k, abq/c; q)_\infty \end{array} \right) = \frac{(-c)^n q^n(a)_n}{(c; q)_n} \Phi_S \left( \begin{array}{c}
a_1, \ldots, a_r \\
b_1, \ldots, b_s; q, dD_q \end{array} \right) \left( \begin{array}{c}
(aq^{1-n}/c, ay; q)_\infty \\
(a, aq/c; q)_\infty \end{array} \right).
\]

Substituting (3.12) and (3.16) in equation (3.14), the proof is completed.

If \( n = \infty \) in equation (3.12), we get a generalization for \( q \)-Gauss sum (1.10) as follows:

**Corollary 3.3.5 (Generalization of \( q \)-Gauss sum).** Let \( q \)-Gauss sum be defined as in (1.10), then

\[
\sum_{k=0}^{\infty} \left( \begin{array}{c}
q^n, b; q \end{array} \right)_k \frac{(c/ab)^k}{(q, c; q)_k} \Phi_S \left( \begin{array}{c}
a_1, \ldots, a_r \\
b_1, \ldots, b_s; q, d/ab \end{array} \right) = \frac{(c/ab)^k}{(q, c/ab; q)_k} \Phi_S \left( \begin{array}{c}
a_1, \ldots, a_r \\
b_1, \ldots, b_s; q, d/ab \end{array} \right).
\]
\[
\frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}} \sum_{i,j \geq 0} \frac{W_{i+j}}{(q; q)_i} \frac{W_{i+j}}{(q; q)_j} \frac{(yc/q; q)_i}{(ay; q)_{i+j}} \frac{(aq/c; q)_j}{(q; q)_j} \\
\times \left[ (-1)^{i+j} q^{\binom{i+j}{2}} (dq/c)^i (d/a)^j \right].
\]

- If \( b = \infty \) in equation (3.12), we get a generalization for \( q \)-Chu-Vandermonde sum (1.7) as follows:

**Corollary 3.3.6** (Generalization to \( q \)-Chu-Vandermonde sum (1.7)). Let \( q \)-Chu-Vandermonde sum be defined as in (1.7), then

\[
\sum_{k=0}^{n} \frac{(q^{-n}, a; q)_k}{(q, c; q)_k} (cq^n/a)^k \sum_{i,j \geq 0} \frac{W_{i+j}}{(q; q)_i} \frac{W_{i+j}}{(q; q)_j} \frac{(q^{-n+k}; q)_i}{(q; q)_i} \frac{(yq^{-k}; q)_j}{(q; q)_j} \\
\times \left[ (-1)^{i+j} q^{\binom{i+j}{2}} (dq^n)^i (dq)^j \right] = \frac{(c/a; q)_n}{(c; q)_n} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} W_{i+j} \frac{(yc/q; q)_i}{(aq/c; q)_j} \frac{(q^{1-n}/c, aq/c; q)_j}{(q, q^{1-n}/c; q)_j} \\
\times \left[ (-1)^{i+j} q^{\binom{i+j}{2}} (dq/c)^i (d/a)^j \right].
\]

- If \( b = 0 \) in equation (3.12), we get a generalization for \( q \)-Chu-Vandermonde sum (1.8) as follows:

**Corollary 3.3.7** (Generalization to \( q \)-Chu-Vandermonde sum (1.8)). Let \( q \)-Chu-Vandermonde sum be defined as in (1.8), then

\[
\sum_{k=0}^{n} \frac{(q^{-n}, a; q)_k}{(q, c; q)_k} q^k r_{s+1} \frac{\phi_{r+1}}{\phi_{s+1}} \left( \begin{array}{c}
\alpha_1, \ldots, \alpha_r, yq^{-k} \\
b_1, \ldots, b_s, ay
\end{array} ; q, dq^k \right) \\
= \frac{(c/a; q)_n}{(c; q)_n} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} W_{i+j} \frac{(yc/q; q)_i}{(ay; q)_{i+j}} \frac{(q^{1-n}/c, aq/c; q)_j}{(q, aq^{1-n}/c; q)_j} \\
\times \left[ (-1)^{i+j} q^{\binom{i+j}{2}} (dq/c)^i (d/a)^j \right].
\]  \hspace{1cm} (3.17)

- If \( r = s = 0 \) and \( y = 0 \) in equation (3.17) then using (1.6), we get the following identity:

**Corollary 3.3.8**

\[
\phi_2^2 \left( \begin{array}{c}
q^{-n}, a, 0 \\
c, d
\end{array} ; q, q \right) = \frac{(dq/c; q)_{\infty}}{(d; q)_{\infty}} \frac{(c/a; q)_n}{(c; q)_n} \frac{(q^{1-n}/c, aq/c; q)}{(aq^{1-n}/c, dq/c; q, d)}
\]

- If \( r = 2, s = 1 \) and \( y = 0 \) in equation (3.17), we get Theorem 17 obtained by Li and Tan [9] (equation (1.22)).
• If \( r = 2, s = 1, y = 0 \) and setting \( a_1 = q^{-N} \) in equation ??, then using equations (1.1) and (1.7), we get Theorem 3.1 obtained by Zhang and Yang [15] (equation (1.19)).

• If \( r = 2, s = 1, y = 0, a_1 = q^{-N} \) and \( a = 1 \) in equation (3.17), we get Corollary 3.2 obtained by Fang [5] (equation (1.20)).

Conclusions

1. Many operators can be obtained by assigning some special values to the generalized \( q \)-operator

\[
\Phi_r \left( a_1, \cdots, a_r, b_1, \cdots, b_s; q, cD_q \right)
\]

2. We generalized some well-known \( q \)-identities, such as Cauchy identity, Heine's transformation formula and the \( q \)-Pfaff-Saalschütz summation formula.
References


[13] H.L. Saad and R.H. Jaber, Application of the Operator $\phi\left(a, b, c; d, e; x, y|q\right)$ for the Polynomials $Y_n(a, b, c; d, e; x, y|q)$, *TWMS J. App. and Eng. Math.*, (2020), accepted.


$$q$$-

تطبيقات المؤثر $$_r \Phi_\nu$$ في المتتابقات.

حسام لوتي سعد
حسن حميل حسن

قسم الرياضيات ، كلية العلوم ، جامعة البصرة ،
البصرة ، العراق

المستخلص:

في هذا البحث، أنشأنا المؤثر العام $$_r \Phi_\nu$$، ثم وجدنا بعض متتابقاته التي سيتم استخدامها لتقسيم بعض متتابقات $q$-

لمثل متتابقة كوشي ، وصيغة تحويل هاين ، وصيغة جمع باف- سلسوتس. من خلال إعطاء قيم خاصة للمعلمات في المتتابقات التي حصلنا عليها ، تم الحصول على بعض النتائج الجديدة و/أو تم إعادة برهان البعض الآخر.