Feedback Linearization of Multi-Input Nonlinear Differential Algebraic Control Systems

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Abstract

The problem of feedback linearization of index one multi-input nonlinear differential algebraic control systems via feedback transformations is addressed. Although necessary and sufficient geometric conditions for this problem have been provided in the early 2000. A complete solution to the feedback linearization problem is provided by defining an algorithm allowing to compute explicitly the linearizing feedback coordinate for index one multi-input nonlinear differential algebraic control systems without solving the partial differential equations. The algorithm consists of steps (the dimension of the system).

Keywords: Feedback linearization, Differential algebraic control, Control systems.

1. Introduction

1. INTRODUCTION

The problem of transforming a nonlinear differential algebraic control system (NDACS)

\[
\Sigma: \begin{cases} 
\dot{x} = f(x,z) + g_1(x,z)u_1 + \ldots + g_m(x,z)u_m \\
0 = \sigma(x,z) 
\end{cases}
\]

into the linear system

\[
A: \begin{cases} 
\dot{\sigma} = A\sigma + B_1v_1 + \ldots + B_mv_m \\
0 = \sigma(\sigma,z) 
\end{cases}
\]

by a feedback transformation of the form

\[
\Gamma: \begin{cases} 
\sigma = \phi(x,z), \quad (x,z) \in M \\
u = \alpha(x,z) + \beta(x,z)v 
\end{cases}
\]

is called feedback linearization problem to the system (1). The linearization problem of nonlinear differential algebraic control system is an important one and has been studied sparsely. Some investigation have been carried out McClamroch et al. with constrained mechanical systems (1)-(2) and also by Kaprielian et al. with an AC/DC power system model (3)-(4). Their approaches consist of using transformations to obtain a state realization (state space representation) of the nonlinear descriptor system and then apply differential geometry for linearization. For single-input nonlinear differential algebraic control systems (5) have defined

\[
F(x,z) = \begin{pmatrix} I_n \\
-(\frac{\partial \sigma}{\partial z})^{-1}\frac{\partial \sigma}{\partial x} 
\end{pmatrix}
\]

where (I_n) is an n x n identity matrix) and deal with the index one NDAE locally as the following nonlinear control,

\[
\begin{pmatrix} \dot{x} \\
\dot{z} 
\end{pmatrix} = F(x,z)f(x,z) + F(x,z)g(x,z)u
\]

to study the exact feedback linearization for this class of NDAS. On the other hand, C. Chen et al. used the ideas of differential geometric control theory to define M.
derivative and $M$ bracket in order to investigate the necessary and sufficient geometric conditions for exact feedback linearization of index one single-input nonlinear differential algebraic control systems (6). The problem of feedback linearization is solved if and only if
\[
(F'1) \iff \text{rank} \left( g, \text{Mad}_f g, \ldots, \text{Mad}^{n-1}_f g \right) = n
\]
\[
(F'2) \quad \text{vector field sets } \Delta = \left\{ g, \text{Mad}_f g, \ldots, \text{Mad}^{n-2}_f g \right\}
\]
are involutive in $(x,z) \in M$. Although, the conditions $(F'1)$ and $(F'2)$ provide a way of testing whether a given system is feedback linearizable but they offer little on how to find the linearizing change of coordinates $\varphi(x,z)$ except by solving a systems of partial differential equations (PDEs) which is, in general, not straightforward. For the problem of feedback linearization of single-input nonlinear differential algebraic control systems, Ayad and Nada (7),(8) provide a complete solution by defining an algorithm that allows to compute explicitly the linearizing state coordinates and feedback for index one nonlinear differential algebraic control systems. Each algorithm is performed using a maximum of $n - 1$ steps ($n$ being the dimension of the system). The objective of this paper is to provide an algorithm giving linearizing feedback coordinates for index one multi-input nonlinear differential algebraic control systems without solving the partial differential equations. The algorithm based on Frobenius Theorem.

2. Notations and Preliminaries
Consider the index one multi-input nonlinear differential algebraic control systems NDACS (1)

\[
\begin{aligned}
\dot{x} &= f(x,z) + g_1(x,z)u_1 + \cdots + g_m(x,z)u_m + \sigma(x,z) \\
\Sigma : & \quad 0 = \sigma(x,z)
\end{aligned}
\]

where
\[
x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n, z = (z_1, \ldots, z_p)^T \in \mathbb{R}^p
\]
and $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$. Also
\[
f(x,z) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n,
\]
\[
g(x,z) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p
\]
and
\[
\sigma(x,z) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}
\]
are smooth vector fields. and assume that its linear system
\[
\begin{aligned}
\overset{\cdot}{\sigma} &= A\sigma + Bu \\
\Lambda : & \quad A\sigma + B_1u_1 + \cdots + B_mu_m = 0
\end{aligned}
\]
is controllable, that is, there exist positive integers $r_1 \geq 1, \ldots, r_m \geq 1$ with
\[
r_1 + \cdots + r_m = n
\]
such that
\[
\text{dim span } \{ A^k B_i, 0 \leq k \leq r_i - 1, 1 \leq i \leq m \} = n.
\]

Define the coordinates
\[
x_k = \left( (x^1_k)^T, \ldots, (x^m_k)^T \right)^T
\]
on\[
\mathbb{R}^n = \mathbb{R}^n \times \cdots \times \mathbb{R}^n,
\]
where for any $1 \leq i \leq r$ we set $x^i_k = (x^i_{k,1}, \ldots, x^i_{k,m})^T$ and we put
\[
\dot{x}_{k,i} = \left( x^1_{k,i}, \ldots, x^i_{k,i}, \ldots, x^2_{k,i}, \ldots, x^m_{k,i} \right)^T
\]

Let the system $\Sigma$ be denoted in the coordinates $x_k$ by $\Sigma_k$
\[
\begin{aligned}
\dot{x}_k &= f_k(x_k,z) + g_{k1}(x_k,z)u_1 + \cdots + g_{km}(x_k,z)u_m + \sigma(x_k,z) \\
\Sigma_k : & \quad 0 = \sigma(x_k,z)
\end{aligned}
\]

and for any $1 \leq i \leq m$ the $i$th subsystem $\Sigma^i_k$ by
\[
\begin{aligned}
\dot{x}^i_k &= f^i_k(x_k,z) + g^i_{k1}(x_k,z)u_1 + \cdots + g^i_{km}(x_k,z)u_m + \sigma(x_k,z) \\
\Sigma^i_k : & \quad 0 = \sigma(x_k,z)
\end{aligned}
\]
For any \(1 \leq i \leq m\) and any \(1 \leq k \leq r\) we define \(A^k_i\) in the following way: for any \(x = (x_1,\ldots,x_n)^T\) we have
\[
A^k_i x = (0,\ldots,0,x_{k+2},\ldots,x_r,0)^T
\]
that is, \(A^k_i\) is the matrix \(A_i\) with the entries in the first \(k\) rows being zeros.

**Definition 2.1:**

The minimum number of times that all or part of the constraint equation must be differentiated with respect to time in order to solve for \(z^j\) as a continuous function of \(x\) and \(z\) is the index of the nonlinear differential algebraic system (1).

**Definition 2.2:**

Let \(f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n\) be a smooth vector field and \(w : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}\) a smooth function. The \(M\) derivative of \(w\) along \(f\) is a function \(\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}\), written \(M_f w\) and defined as \(M_f w = E(w)f\), where
\[
E(w) = \frac{\partial w}{\partial x} - \frac{\partial (\partial \sigma)}{\partial z} \frac{\partial \sigma}{\partial x} \quad \text{if } w \quad \text{is differential } k \text{ times along } f,
\]
and \(M^k_f w = M_f \left( M^{-1}_f w \right) \) with \(M_f^0 w = w\).

**Definition 2.3:**

Given two smooth vector fields \(f(x,z)\) and \(g(x,z)\), both are defined on \(\mathbb{R}^n\) then the \(M\) bracket is defined as follows:
\[
\text{Mad}_{f(x,z)} g(x,z) = \left[ f(x,z), g(x,z) \right]_M = E(g)f - E(f)g
\]
Repeated \(M\) brackets are denoted as
\[
\text{Mad}^k_f (x,z) g(x,z) = \text{Mad}_f (\text{Mad}^{k-1}_f g),
\]
\[
\text{Mad}^{-1}_f (x,z) g(x,z) = \text{Mad}_f g
\]
and
\[
\text{Mad}^{\ast}(x,z) g(x,z) = g.
\]
Also,
\[
\left[ f(x,z), g(x,z) \right]_M = -\left[ g(x,z), f(x,z) \right]_M
\]
and
\[
\left[ f, g \right]_M w(x,z) = M_f M_g w - M_g M_f w.
\]

**Theorem 2.4:**

Consider the partial differential equation of function \(w(x,z)\) with constraint condition \(\sigma(x,z) = 0\)
\[
E(w)[v_1(x,z) v_2(x,z) \ldots v_3(x,z)] = 0
\]
in which
\[
E(w) = \frac{\partial w}{\partial x} - \frac{\partial (\partial \sigma)}{\partial z} \frac{\partial \sigma}{\partial x}
\]
where
\[
(x,z) \in \mathbb{R}^n \times \mathbb{R}^m, v_i(x,z) (i = 1,2,\ldots,k < n)\]
are linearly independent vector fields. If vector field set
\[
D = \{ v_1(x,z) v_2(x,z) \ldots v_3(x,z) \}
\]
is involutive at \((x,z) = (x^0, z^0)\), then there exist certain \((n-k)\) functions \(w^1(x,z), w^2(x,z), \ldots, w^{n-k}(x,z)\) which satisfy given partial differential equation groups and the vectors
\[
\left[ E_1(w^j) E_2(w^j) \ldots E_n(w^j) \right] (j = 1,2,\ldots,(n-k)),
\]
\[
E_i = \partial / \partial x^i - \sum_{k=1}^m r_k \partial / \partial z^i, i = 1,2,\ldots,n
\]
are linearly independent at \((x^0, z^0)\).

**Theorem 2.5:**

Let \(u\) be a smooth vector field on \(\mathbb{R}^n\), for any integer \(1 \leq k \leq n\) such that \(v_k(0,0) \neq 0\) and \(\omega_k(x,z) = 1/v_k(x,z)\).
The diffeomorphism \(\xi = \varphi(x,z)\), where \(\varphi : M \rightarrow \mathbb{R}^n\) defined by
\[
\varphi_j(x,z) = x_j + \sum_{s=1}^{\infty} \frac{(-1)^s x_k^s}{s!} M^{-1}_{\omega_k \omega_j}(\omega_k(x,z)),
\]
\[
\varphi_k(x,z) = \sum_{s=1}^{\infty} \frac{(-1)^s x_k^s}{s!} M^{-1}_{\omega_k \omega_k}(\omega_k(x,z))
\]
(4)
Satisfies \(\varphi^*(\nu) = \partial / \partial \tilde{z}_i\). Moreover, the diffeomorphism \(\psi(\xi)\), where \(\psi : \mathbb{R}^n \rightarrow M\) defined by
ψ_j(ξ) = ξ_j + ∑_{i=1}^{∞} \frac{e^s_i}{s!} \left[ \sum_{i=0}^{∞} (-1)^i \frac{∂ \xi_i}{∂ ξ_k} M_{i,j} \right] (ν_j)

ψ_k(ξ) = ∑_{i=1}^{∞} \frac{e^s_i}{s!} \left[ \sum_{i=0}^{∞} (-1)^i \frac{∂ \xi_i}{∂ ξ_k} M_{i,j} \right] (υ_k)

(5)
is the inverse of ξ = ϕ(x,z).

where

∂_ξi = ∂_ξk, h = ∂_ξk, ..., h = ∂_ξk

and C'_i = \frac{s!}{i!(s-i)!}, i ≥ 2.

3. MAIN RESULTS

Definition 3.1
The index one multi-input nonlinear differential algebraic control systems Σ_k is called (FB)_k -linear form if for 1≤i≤m the ith subsystem Σ'_k decomposes

Σ'_k : [\begin{align*}
\dot{x}'_k &= F'_{k,j} (\hat{x}_{k,k+1}, z) & \text{if } 1 ≤ j ≤ k - 1 \\
\dot{x}'_k &= x'_{k,j+1} & \text{if } k ≤ j ≤ r - 1 \\
0 &= σ(\hat{x}_{k,k+1}, z)
\end{align*}]

(6)

where F'_k (0,0) = 0 and \frac{∂F'_k}{∂ x_{k,j+1}} (0,0) = 0. It follows easily that if Σ_k is (FB)_k -linear, then

Mad_{k'}(g_{k,i}) = A^{j-1}B_i,

1 ≤ j ≤ r - k + 1, 1 ≤ i ≤ m

(7)

A more compact representation of Σ'_k is obtained as

Σ'_k : [\begin{align*}
\dot{x}'_k &= A'_{k,j} x'_k + F'_{k,j} (\hat{x}_{k,k+1}, z) + b_i u_{k,i}, x'_k ∈ \mathbb{R}' \\
0 &= σ(\hat{x}_{k,k+1}, z)
\end{align*}]

(8)

with the last r - k components of F'_{k,j} being identically zero. By extension, a compact notation for Σ_k would be

Σ_k : \begin{align*}
\dot{x}_k &= A_k x_k + F_k (\hat{x}_{k,k+1}, z) + B_i u_1 + ... + B_m u_m \\
0 &= σ(\hat{x}_{k,k+1}, z)
\end{align*}

where

F_k (\hat{x}_{k,k+1}, z) = (F'_k(\hat{x}_{k,k+1}, z))^T, ..., (F'_k(\hat{x}_{k,k+1}, z))^T

Theorem 3.2
Consider the index one multi-input NDACS

Σ_r : \begin{align*}
\dot{\hat{x}}_r &= f_r (x_r, z) + g_r (x_r, z) u_{k_1} \\
& \vdots \\
0 &= σ(x_r, z)
\end{align*}

Assume it is feedback linearizable, that is, satisfies (F′1) and (F′2).

There exists a sequence of explicit feedback transformations

Γ_r = (φ_r, α_r, β_r), ..., Γ_1 = (φ_1, α_1, β_1) giving rise to a sequence of (FB)_k -linear systems

Σ_{r,1}, ..., Σ_0 such that

Σ_{k-1} = Γ_k * (Σ_k) = (φ_k, α_k, β_k) * Σ_k, 1 ≤ k ≤ r

The (FB)_k -linear system Σ_k can be transformed into a (FB)_{k-1} -linear system Σ_{k-1} if and only if for all 1 ≤ i, j ≤ m

(a) \frac{∂^2 f_k (\hat{x}_{k,k+1}, z)}{∂ x_{k,k+1}^j ∂ x_{k,k+1}^j} = 0

(b) \begin{bmatrix}
\frac{∂f_k}{∂ x_{k,k+1}^j} \\
\frac{∂f_k}{∂ x_{k,k+1}^i}
\end{bmatrix} = \begin{bmatrix}
\frac{∂^2 f_k (\hat{x}_{k,k+1}, z)}{∂ x_{k,k+1}^j ∂ x_{k,k+1}^i} \\
\frac{∂^2 f_k (\hat{x}_{k,k+1}, z)}{∂ x_{k,k+1}^i ∂ x_{k,k+1}^j}
\end{bmatrix}

(8)

Thus the composition Γ_1 * ... * Γ_r linearizes the system Σ_r.

Algorithm 3.3
Step r. Consider a feedback linearizable system $\Sigma$ denoted in the coordinates $x = x_r$ by $\Sigma_r$:

$$\dot{x}_r = f_r(x_r, z) + g_{rl}(x_r, z)u_{rl}$$

$$\Sigma_r : \begin{cases} \dot{x}_r = f_r(x_r, z) + g_{rl}(x_r, z)u_{rl} \\ + \cdots + g_{rm}(x_r, z)u_{rm} \\ 0 = \sigma(x_r, z) \end{cases}$$

since $\Sigma_r$ is feedback linearizable, hence the distribution $D_r = \{g_{r1}, \ldots, g_{rm}\}$ involutive, we apply Theorem (2.5) to construct $\xi_r = \phi_r(x_r, z)$ such that $(\phi_r)^* \{g_{r1}, \ldots, g_{rm}\} = \check{\beta}(\xi_r, z) \{\check{\alpha}_{r1}, \ldots, \check{\alpha}_{rm}\}$. Then apply the feedback $\tilde{u}_r = \check{\alpha}_i(\xi_r, z) + \check{\beta}(\xi_r, z)u_r$, where

$$\check{\alpha}_i(\xi_r, z) = (\check{\alpha}_{i1}(\xi_r, z), \ldots, \check{\alpha}_{im}(\xi_r, z))^T$$

is such that $\check{\alpha}_i(\xi_r, z)$ cancels the last component of $(\phi_r)^* f_i$, to bring $\Sigma_r$ into

$$\Sigma_r = (\phi_r)^* f_i : \begin{cases} \dot{\xi}_r = f_i(\xi_r, z) + g_{ri}(\xi_r, z)u_{ri} \\ + \cdots + g_{rm}(\xi_r, z)u_{rm} \\ 0 = \sigma(\xi_r, z) \end{cases}$$

where $f_i = (\phi_r)^* f_i = A_i^{-1} \check{f}_i(\xi_r, z)$ and $g_{ri} = \check{\alpha}_{ri} = B_i$.

Each subsystem is of the form

$$\Sigma_i^r = (\phi_r)^* f_i : \begin{cases} \dot{\xi}_r = \check{f}_i(\xi_r, z) + B_i \tilde{u}_ri \\ 0 = \sigma(\xi_r, z) \end{cases}$$

with the $r$th component of $\check{f}_i$ zero, i.e., $\check{f}_i(\xi_r, z) = 0$. To normalize the $(r-1)$th component of $\check{f}_i$, we apply the push-forward change of coordinates $x_{r-1} = \phi_{r-1}(\xi_{r-1}, z)$ given by

$$x_{r-1} = \phi_{r-1}(\xi_{r-1}, z) = \begin{cases} x_{r-1j} = \phi_{r-1j}(\xi_{r-1}, z) = \xi_{r-1j}, & 1 \leq j \leq r-1 \\
 x_{r-1r} = \phi_{r-1r}(\xi_{r-1}, z) = \check{f}_i(\xi_r, z) \\
 \end{cases}$$

followed by a feedback

$$u_{r-1} = (u_{r-11}, \ldots, u_{r-1m})^T$$

$$u_{r-1}^i = M_i \check{\phi}_i(\xi_r, z) + \sum_{j=1}^m M_{ij} \check{\phi}_j(\xi_r, z) \tilde{u}_rj,$$

$$1 \leq i \leq m$$

The composition

$$x_{r-1} = \phi_r(\xi_r, z) = \phi_{r-1} \circ \phi_r(x_r, z)$$

in terms of $u_r$, form a transformation $\Gamma$ such that

$$\Gamma_r : \Sigma_r = \Sigma_{r-1}$$

$$\Sigma_{r-1} : \begin{cases} \dot{x}_{r-1} = f_{r-1}(x_{r-1}, z) + g_{r-1l}(x_{r-1}, z)u_{r-1l} \\ + \cdots + g_{r-1m}(x_{r-1}, z)u_{r-1m} \\ 0 = \sigma(x_{r-1}, z) \end{cases}$$

which is $(FB)_{k-1}$–linear as it satisfies (7) with $k = r-1$.

Step k. Assume that $\Sigma$, has been transformed, via explicit coordinates changes and feedback, into a $(FB)_k$–linear system

$$\Sigma_k : \begin{cases} \dot{x}_k = f_k(x_k, z) + g_{ki}(x_k, z)u_i \\ + \cdots + g_{km}(x_k, z)u_m \\ 0 = \sigma(x_k, z) \end{cases}$$

with $f_k(x_k, z) = A_k x_k + F_k(\hat{x}_{k+k-1}, z)$ and $g_{ki}(x_k, z) = B_i$ for all $1 \leq i \leq m$. Since $\Sigma$, (hence $\Sigma_k$) is feedback linearizable, then condition $(F'2)$ is satisfied, implying in particular,

$$\left[ Mad_i^k(g_{ki}), Mad_j^k(g_{kj}) \right]$$

$$= \sum_{l=0}^{s} \sum_{p=1}^{m} \Theta_p^i(\hat{x}_{k+k-1}, z) Mad_j^k(g_{kp})$$

$1 \leq i, j \leq m$ and $s \geq t \geq 0$, where $\Theta_p^i(\hat{x}_{k+k-1}, z)$ are functions of the indicated variables. Since (7) of definition (3.1) holds, setting $s = r-k$ and $t = r-k-1$ implies
using (7) we see that all coefficients $\Theta'_p$ are zero, i.e., the vector field $f_k(\hat{x}_{k+1},z)$ decomposes uniquely as

$$f_k(\hat{x}_{k+1},z) = A^k x_k + \ddot{F}_k(\hat{x}_{k+1},z) + \sum_{i=1}^m x^i_k \ddot{g}_k(\hat{x}_{k+1},z)$$

where

$$\hat{x}_{k+1} = (x^1_{k+1}, \ldots, x^m_{k+1}, x^2_k, \ldots, x^m_k)$$.

By Theorem 2.5 we construct a change of coordinates $\xi_k = \varphi_k(x_k,z)$ that rectifies the involutive distribution $\tilde{D}_k = \text{span}\{\ddot{g}_k(\hat{x}_{k+1},z)\}$. Then we define a push-forward change of coordinates followed by an appropriate feedback transformation whose composition with $\xi_k = \varphi_k(x_k,z)$ yields a transformation $\Gamma_k$ that maps $\Sigma_k$ into $\Sigma_{k-1}$.

**Example 3.4**

Consider the index one multi-input nonlinear differential algebraic control systems

$$\Sigma_3 : \begin{cases} \dot{x}_3 = f_3(x_3, z) + g_{31}(x_3, z)u_1 \\ + g_{32}(x_3, z)u_2 \\ 0 = \sigma(x_3, z) \end{cases}$$

Defined in the coordinates

$$x_3 = (x_{31}, \ldots, x_{35})^T \in \mathbb{R}^5$$

$$\Sigma_3 : \begin{cases} \dot{x}_{31} = x_{32}(1 + x_{33}) \\ \dot{x}_{32} = x_{33}(1 + x_{31}) - x_{32}u_{31} \\ \dot{x}_{33} = x_{33} + x_{35} + z + (1 + x_{33})u_{31} \\ \dot{x}_{34} = x_{35} + z \\ \dot{x}_{35} = u_{32} \\
0 = x_{31}^2 - z \end{cases}$$

where

$$\begin{bmatrix} \frac{\partial \sigma}{\partial z} \end{bmatrix}^{-1} = -1$$

$$\begin{bmatrix} \frac{\partial \sigma}{\partial x} \end{bmatrix} = (2x_{31}, 0, 0, 0, 0)$$

$$\begin{bmatrix} \frac{\partial \sigma}{\partial x} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial \sigma}{\partial x} \end{bmatrix} = (-2x_{31}, 0, 0, 0, 0)$$

$$f_3(x_3, z) = \begin{pmatrix} x_{32}(1 + x_{33}) \\ x_{33}(1 + x_{31}) + x_{35} + z, x_{35} + z \\ 0 \end{pmatrix}$$

$$g_{31}(x_3, z) = (0, -x_{32}, (1 + x_{33}), 0, 0)^T$$

$$g_{32}(x_3, z) = (0, 0, 0, 1)^T$$

To rectifies the distribution $D_3 = \text{span}\{g_{31}, g_{32}\}$ we look for a change of coordinates $y = \varphi_k(x_3, z)$. Apply Theorem (2.5) to $v = g_{31}(x_3, z)$ with $n = 5$ and $\sigma_3 = (1 + x_{33})^{-1}$.

Thus

$$\sigma_3 v = \begin{pmatrix} 0, -x_{32}(1 + x_{33}), 1, 0, 0 \end{pmatrix}^{-1}$$

Since $M_\sigma^{-1} v_s = 0$ for all $s \geq 1$ we get

$$y_1 = x_{31} + \sum_{s=1}^\infty T^s \int_{y_1}^T \sum_{t=1}^\infty M_\sigma^{-1} \sigma_3 v_s(x_3, z)$$

On the other side $v_2(x_3, z) = -x_{32}$ implies

$$M_\sigma^{-1} \sigma_3 v_2 = 2x_{32}(1 + x_{33})^{-2}$$

$$M_\sigma^{-1} \sigma_3 v_2 = -6x_{32}(1 + x_{33})^{-3}$$

which gives

$$M_\sigma^{-1} \sigma_3 v_2 = (-1)^s s! x_{32}(1 + x_{33})^{-s}$$.

Thus
\[ y_2 = x_{32} + \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} M_{s, \sigma}^{-1}(\sigma_j \sigma_2)(x_3, z) \]
\[ = x_{32}(1 + x_{33}) \]
Notice that \( M_{\sigma, \sigma}^{-1} \sigma_3 = -(1 + x_{33})^{-2} \), \( M_{\sigma, \sigma}^{-2} \sigma_3 = 2(1 + x_{33})^{-3} \) which gives \( M_{\sigma, \sigma}^{-1} \sigma_3 = -(s - 1)!(1 + x_{33})^{-s} \). Thus
\[ y_3 = \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} M_{s, \sigma}^{-1}(\sigma_3)(x_3, z) \]
\[ = \sum_{s=1}^{\infty} \frac{1}{s} \left( \frac{x_3}{1 + x_{33}} \right)^s \]
\[ = \ln(1 + x_{33}) \]
We apply the change of coordinates
\[ y = \phi_3(x_3, z) = \begin{cases} 
  y_1 = x_{31} \\
  y_2 = x_{32}(1 + x_{33}) \\
  y_3 = \ln(1 + x_{33}) \\
  y_4 = x_{34} \\
  y_5 = x_{35} \\
  0 = x_{31} - z 
\end{cases} \]
Whose inverse is given by
\[ x_3 = \phi_3^{-1}(y, z) = \begin{cases} 
  x_{31} = y_1 \\
  x_{32} = y_2 e^{-y_3} \\
  x_{32} = e^{y_3} - 1 \\
  x_{32} = y_4 \\
  x_{32} = y_5 \\
  0 = y_1^2 - z 
\end{cases} \]
To transform the original system into
\[ \dot{y}_1 = y_2 \]
\[ \dot{y}_2 = (1 + y_1)e^{y_3}(e^{y_3} - 1) + y_2 e^{-y_3}(y_1 + y_5 + z) \]
\[ \dot{y}_3 = e^{-y_3}(y_1 + y_5 + z) + u_{31} \]
\[ \dot{y}_4 = y_5 + z \]
\[ \dot{y}_5 = u_{32} \]
\[ 0 = \sigma(y, z) = y_1^2 - z \]
\[ \left( \frac{\partial \sigma}{\partial z} \right)^{-1} = -1 \]
\[ \frac{\partial \sigma}{\partial y} = (2 y_1 0 0 0 0) \]
\[ \left( \frac{\partial \sigma}{\partial z} \right)^{-1} \frac{\partial \sigma}{\partial y} = (-2 y_1 0 0 0 0) \]
This is (FB)-form and can be put into
\[ x_{21} = \phi_1(y, z) = y_1 \]
\[ x_{22} = \phi_2(y, z) = y_2 \]
\[ x_{23} = \phi_3(y, z) = y_3 \]
\[ x_{24} = \phi_4(y, z) = y_4 \]
\[ x_{25} = \phi_5(y, z) = y_5 \]
\[ 0 = y_1^2 - z \]
\[ u_{21} = M_{f_1} \phi_3(y, z) + M_{f_2} \phi_2(y, z) u_{31} + M_{f_3} \phi_1(y, z) u_{32} \]
\[ = \left[ e^{y_3} \left( e^{y_3} - 1 \right) + y_2 e^{-y_3} + 2 y_1 y_2 e^{-y_3} \right] y_2 \]
\[ + 2 \left[ (1 + y_1) \left( e^{y_3} - 1 \right) (y_1 + y_5 + y_1^2) \right] \]
\[ + (1 + y_1) e^{y_3} (y_1 + y_5 + y_1^2) + y_2 e^{-y_3} u_{32} \]
\[ + \left[ (1 + y_1) e^{y_3} \left( (e^{y_3} - 1) + e^{y_3} \right) + y_2 e^{-y_3} (y_1 + y_5 + y_1^2) \right] u_{31} \]
\[ u_{22} = M_{f_1} \phi_3(y, z) + M_{f_2} \phi_2(y, z) u_{31} \]
\[ + M_{f_3} \phi_1(y, z) u_{32} = 2 y_1 y_2 + u_{32} \]
The composition \( x_2 = \phi \circ \phi_3(x_3, z) \) gives
\[ x_{21} = x_{31} \]
\[ x_{22} = x_{32}(1 + x_{33}) \]
\[ x_{23} = (1 + x_{31}) x_{35}(1 + x_{33}) + x_{32}(x_{31} + x_{35} + x_{31}^2) \]
\[ x_{24} = x_{34} \]
\[ x_{25} = x_{35} + x_{31}^2 \]
Brings \( \Sigma_3 \) into linear form
\[
\begin{align*}
\dot{x}_{21} &= x_{22} \\
\dot{x}_{22} &= x_{32} \\
\dot{x}_{23} &= u_{21} \\
\dot{x}_{24} &= x_{21} \\
\dot{x}_{25} &= u_{22}
\end{align*}
\]

\[\Sigma_2 : \]

References