STABILITY OF ITERATIVE PROCEDURES
FOR HYBRID MAPS IN b-METRIC SPACE

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Abstract.
In this paper, we study the existence of coincidences and fixed points of hybrid contractions maps on more general setting than metric spaces.

The same utilize to discuss the problem of stability of iterative procedures in generalized Hausdorff metric space.


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1. INTRODUCTION
Let \((X,d)\) be a metric space and an operator \(T: X \rightarrow \text{CL}(X)\) where \(\text{CL}(X)\) the collection of all nonempty closed subsets of \(X\). Problem of existence of coincidence and fixed points of \(T\) lead to an iteration procedure \(x_{n+1} \in f(T, x_n)\). However, in computation, an approximate sequence \(\{y_n\}\) is applied instead of \(\{x_n\}\). This leads to a concept of stability of iteration procedure with respect to \(T\).

The first stability result in metric space was proved by Ostrowski (1967). For
any $x_0 \in X$, we put $x_{n+1} = f(T,x_n)$, $n = 1,2,\ldots$. \hfill (1)

Let $\{x_n\}$ be a sequence converges to a fixed point $u \in X$ of $T$. Let
$\{y_n\} \subseteq X$ be an arbitrary sequence, and set
$\varepsilon_n = d(y_n, Ty_n)$, $n = 0,1,2,\ldots$. \hfill (2)

The iteration procedure (1) is said to be $T$-stable (cf Czerwik 1998) provided
that $\lim_{n \to \infty} \varepsilon_n = 0$ implies that $\lim_{n \to \infty} y_n = u$.

This subject was developed by Harder and Hicks 1988a, b, Rhoades 1993,
Osilike 1995, Osilike 1996 and Berinde 2002 (see also Ciric 1972, Ciric 1974,

Czerwik 1997 and Czerwik et al. 2002 has been extended Ostrowski’s
classical theorem for the stability of iterative procedures for multi-valued maps
in $b$-metric spaces. The purpose of this paper is to study the stability of iteration
process for hybrid contraction mappings involving single-valued and multi-valued
maps in $b$-metric spaces.

2. PRELIMINARIES

Consistent with Czerwik 1998 and
Singh & Hashim 2005, we use the
following notations and definitions.

Definition 2.1: Czerwik 1998.

Let $X$ be a nonempty set and $s \geq 1$ be a
given real number. A function
$d : X \times X \to \mathbb{R}^+$ is a $b$-metric provided
that for all $x, y, z \in X$,
\[
d(x, y) = 0 \quad \text{iff} \quad x = y, \quad (3)
\]
\[
d(x, y) = d(y, x), \quad (4)
\]
\[
d(x, z) < s [d(x, y) + d(y, z)], \quad (5)
\]
The pair $(X, d)$ is called a $b$-metric spaces.

Notice that the class of $b$-metric spaces is
effectively larger than that of metric spaces.
The following example shows that a $b$-
metric on $X$ need not a metric on $X$ (see
also Singh & Prasad 2008).

Example 2.1: Let $X = \{x_1, x_2, x_3\}$ and
d : $X \times X \to \mathbb{R}^+$ such that
\[
d(x_1, x_2) = a \geq 2, \quad d(x_1, x_3) = d(x_2, x_3) = 1 \quad \text{and}
\]
\[
d(x_n, x_n) = 0, \quad d(x_n, x_k) = d(x_k, x_n),
\]
\[
d(x_n, x_k) = \frac{a}{2} [d(x_n, x_k) + d(x_k, x_n)], \quad n, k, i = 1, 2, 3
\]

Then $(X, d)$ is a $b$-metric space.

$CL(X) = \{ A : A$ is nonempty closed
subset of $X \}$
If the maximum exists
otherwise
\[
H( A , B ) = \max \left\{ \sup_{x \in A} D( x , B ) , \sup_{y \in B} D( y , A ) \right\}
\]

If \( \varepsilon_n = 0 \) implies that \( \lim_{n \to \infty} y_n = p \) then the iteration process defined in (*) is said to be \( T \)-stable or stable with respect to \( T \).

Recall that this definition for a single-valued operator is due to Harder & Hicks 1988a, b.

The following definition of stability general iterative procedure (9) for the coincidence point equation \( fx = Tx \) is due to Singh et al. 2008 when \( S \) and \( T \) are single-valued maps.

**Definition 2.3** [Singh et al. 2008]:

Let \( ( X , d ) \) be a \( b \)-metric space and \( Y \subseteq X \). Let \( T : Y \to CL( X ) \), \( S : Y \to X \), \( T( Y ) \subseteq S( Y ) \) and \( z \) a coincidence point of \( T \) and \( S \), that is \( Sz \in Tz = u \). For any \( x_o \in Y \), let the sequence \( \{ Sx_n \} \) generated by the general iterative procedure.

\[
Sx_{n+1} \in f( T , x_n ) \quad n = 1 , 2 , \\
\]

Converge to an element \( u \in X \). Let \( \{ Sy_n \} \subseteq X \) be an arbitrary sequence and let \( \varepsilon_n = H( Sy_{n+1} , f( T , y_n ) ) \), \( n = 0 , 1 , 2 \).
Then the iterative procedure \( f(T,x_n) \) is 
\((S,T)\), stable or stable with respect to 
\((S,T)\) if
\[ \lim_{n \to \infty} \epsilon_n = 0 \]
implies that
\[ \lim_{n \to \infty} S y_n = 0. \]

We remark that the iterative procedure (9) reduce to
\[ x_{n+1} = f(T,x_n) \quad n = 1, 2, \ldots \]
when \( Y = X \) and \( S \) is the identity map on \( X \). The general definition of stability of iterative procedures for general fixed equation due to Czerwik et al. 2002 and Singh et al. 2005.

Consider the following conditions for
\( T : Y \to CL(X) \) and \( S : Y \to X \) for all \( x, y \in Y \), where \( 0 < q < 1 \) and \( X \) is a metric space;
\[ H(Tx,Ty) \leq q d(Sx,Sy); \quad (10) \]
\[ H(Tx,Ty) \leq q \max\{ d(Sx, Sy), D(Sx, Tx), D(Sy, Ty), \frac{l}{2} [ D(Sx, Ty) + d(Sy, Tx) ] \}; \quad (11) \]
\[ H(Tx,Ty) \leq q d(Sx, Sy) + LD(Sx, Tx). \quad 0 < q < 1, L \geq 0 \quad (12) \]

Notice that (10) is included in (11) while (11) itself is called a generalized hybrid contraction and when \( S = id \) identity map on \( X \) (11) is called a generalized multi-valued contraction. These conditions with \( X = Y \), \( S = id \), the identity map on \( X \) and \( T \) is a single valued mapping are compared in Rhoades 1977.

Lemma 2.3: see also (Rhoades 1993, Czerwik et al. 2002).

Let \( \{ \epsilon_n \} \) be a sequence of non negative real numbers.

Let \( s_n = \sum_{i=0}^{n} k^{n-i} \epsilon_i \) where \( 0 \leq k < 1 \).

Then \( \lim_{n \to \infty} \epsilon_n = 0 \) iff \( \lim_{n \to \infty} s_n = 0 \). \( (13) \)

Now we can present the following fixed point theorem (see Singh et al. 2005):

**Theorem 2.1:** (Singh et al. 2005):

Let \( X \) be a complete b-metric space and \( T \) a generalized multi-valued contraction on \( X \). Then

(i) For every \( x_o \in X \) there exists an orbit \( \{ x_n \} \) of \( T \) at \( x_o \) and \( u \in X \) such that \( \lim_{n \to \infty} x_n = u \).

(ii) The point \( u \) is fixed point of \( T \), i.e. \( u \in Tu \).

3. Main Results

Theorems 3.1:

Let \( (X,d) \) be a b-metric and 
\[ T : Y \to CL(X), \quad S : Y \to X \] such that 
\[ T(Y) \subseteq S(Y) \] and (11) holds with 
\[ qs < 1 \] and 
\[ \alpha s < 1 \] where 
\[ \alpha = \max\{ k, \frac{ks}{2 - qs} \} < 1 \] and one of
$TY$ or $SY$ is a complete subspace of $X$, then $T$ and $S$ have a coincidence point, i.e. there exists a $v \in Y \ni Sv \in Tv$.

**Proof:**

Let $x_o \in Y$ since $T(Y) \subseteq S(Y)$, choose $x_i$, so that $y_i = Sx_i \in Tx_o$.

In general choose $x_{n+i}$ so that $y_{n+i} = Sx_{n+i} \in Tx_n$. Let $\lambda$ be a real number such that $0 < \lambda < 1$. By lemma (2.2) and condition (11),

$$d(y_a, y_{a+l}) \leq q^{-l} H(Tx_{n+l}, Tx_a)$$

$$\leq q^{-l} q \max \{ d(Sx_{n+l}, Sx_a), D(Sx_{n+l}, Tx_a), \frac{1}{2}[D(Sx_{n+l}, Tx_a) + D(Sx_a, Tx_{n+l})] \}$$

$$d(y_a, y_{a+l}) \leq k \max \{ d(y_a, y_a), d(y_{a-l}, y_a), d(y_a, y_{a+l}) \}$$

$$d(y_a, y_{a+l}) = \frac{1}{2} [d(y_{a-l}, y_{a+l}) + d(y_a, y_a)]$$

where $k = q^{l-\lambda}$.

Since $k < 1$ the relation

$$d(y_{n-i}, y_{n+i}) \leq kd(y_n, y_{n+l}) \text{ implies }$$

$$d(y_{n-i}, y_{n+i}) = 0.$$ 

We may assume that $y_n \neq y_{n+i}$ and

$$d(y_a, y_{a+l}) \leq k \max \{ d(y_{a-l}, y_a), \frac{1}{2} d(y_{a-l}, y_{a+l}) \}$$

so

$$d(y_n, y_{n+l}) \leq kd(y_{n-l}, y_n) \quad \text{ or }$$

else

$$d(y_n, y_{n+l}) \leq \frac{ks}{2} d(y_{n-l}, y_n) + d(y_n, y_{n+l})$$.

so $d(y_n, y_{n+l}) \leq \frac{ks}{2 - ks} d(y_{n-l}, y_n)$

This yield

$$d(y_n, y_{n+l}) \leq \max \{ \frac{ks}{2 - ks} d(y_{n-l}, y_n) \} \quad n = 1, 2, \ldots.$$ 

where $\max \{ k, \frac{ks}{2 - ks} \} < 1$.

Therefore, $\{ y_n \}$ is a Cauchy sequence and its sub sequence $\{ y_{2n} \}$ has a limit in $SY$ call it $u$. Let $v \in S^{-1} u$. Then $Sv = u$.

Notice that the subsequence $\{ y_{2n+l} \}$ also converge to $u$ by (11)

$$D(Tv, Sv) \leq H(Tv, Tx_{2n+l})$$

$$\leq q \max \{ D(Sv, Sx_{2n+l}), D(Tv, Tv), D(Sx_{2n+l}, Tx_{2n+l}) \}$$

$$\frac{1}{2} [D(Tv, Tx_{2n+l}) + D(Sx_{2n+l}, Tv)]$$

$$\leq q \max \{ D(Sv, y_{2n+l}), D(Tv, y_{2n+l}), d(y_{2n+l}, y_{2n+l}), \frac{1}{2} [D(Sv, y_{2n+l}) + D(y_{2n+l}, Tv)] \}.$$ 

Making $n \to \infty$ this obtains

$$D(Tv, Sv) \leq q D(Sv, Tv),$$

and $Sv \in Tv$.

We remark that theorem (3.1) with $S = Id$ (the identity map) is the main result of Singh et al. 2005.

If $T$ is a single-valued map in theorem 3.1 with $S = I$ is also the main result of Ciric 1972.

**Theorem 3.2:** let $(X, d)$ be a b-metric and $Y \subseteq X$ let $T : Y \to CL(X)$. $S : Y \to X$ and $SY$ or $TY$ is a complete subspace of $X$. 

Let \( z \) be a coincidence point of \( T \) and \( S \), i.e., \( u = Sz \in Tz \). For any \( x_o \in Y \), let the sequence \( \{ Sx_n \} \), generated by \( Sx_{n+t} \in T x_n \) converge to \( u \). Let \( \{ Sy_n \} \subseteq X \) and define \( \varepsilon_n = H(Sy_{n+1}, Ty_n), \quad n = 0, 1, 2, \ldots \).

If \( S \) and \( T \) satisfy (11) for all \( x, y \in Y \) and \( k = sq < 1 \), then

\[
d(u, Sy_{n+1}) \leq sd(u, Sx_{n+1}) + s(\alpha)^{n+1}d(Sx_n, Sy_n) + s\varepsilon_n + \varepsilon_{n+1},
\]

(14)

\[\alpha = s^2 q / 1 - s^2 q\]

\[\lim_{n \to \infty} Sy_n = u \iff \lim_{n \to \infty} \varepsilon_n = 0 \quad (15)\]

**Proof:** From (11) for any \( x, y \in Y \), one of the following holds:

\[H(Tx, Ty) \leq qd(Sx, Sy), \quad H(Tx, Ty) \leq qd(Sx, Ty) \leq qH(Sx, Ty); \quad H(Tx, Ty) \leq qd(Sy, Ty) \leq qH(Sy, Ty) \leq qH(Sy, Ty) + H(Sx, Ty)\]

\[\leq qsd(Sy, Sx) + q[s H(Sx, Tx) + H(Tx, Ty)] \leq qsd(Sy, Sx) + q2s H(Sx, Tx) + q3s H(Tx, Ty)\]

i.e.,

\[H(Tx, Ty) \geq \frac{q^2}{2^3} d(Sy, Sx) + \frac{q3s^2}{1 - q3s} H(Sx, Tx)\]

\[\frac{k}{1 - ks} d(Sx, Sx) + \frac{ks}{1 - ks} H(Sx, Tx)\]

\[\frac{q}{2} H(Sx, Ty) + H(Sy, Tx)\]

\[\frac{q}{2} H(Sx, Ty) + H(Sx, Ty) + H(Sx, Ty) + H(Sx, Ty)\]

\[\frac{q}{2} \{2H(Sx, Ty) + H(Sy, Ty) + H(Tx, Ty)\}\]

i.e,

\[H(Tx, Ty) \leq \frac{q^2}{2^3} \{2H(Sx, Ty) + d(Sx, Sy)\}
\]

\[\leq \frac{k}{2} d(Sx, Sy) + \frac{2k}{2 - k} H(Sx, Ty)\]

Therefore, in all the cases we get,

\[H(Tx, Ty) \leq \alpha d(Sx, Sy) + s\alpha H(Sx, Tx)\]

Where \( \alpha = \frac{s^2 q}{1 - s^2 q} \)

Then by virtue of (11) we obtain, for any non-negative integer \( n \).

\[d(Sx_{n+1}, Sy_{n+1}) \leq d(Sx_n, Sy_n) \leq d(Sx_n, Ty_n) + d(Ty_n, Sy_n) + \varepsilon_n\]

\[\leq d(Sx_n, Ty_n) + s\alpha H(Sx_n, Ty_n) + \varepsilon_n\]

\[\leq s\alpha d(Sx_n, Sx_n) + s\alpha H(Sx_n, Ty_n) + s\varepsilon_n\]

\[\leq s\alpha d(Sx_n, Sx_n) + s\alpha H(Sx_n, Ty_n) + s\alpha d(Sx_n, Ty_n)\]

\[\leq s\alpha d(Sx_n, Sy_{n+1}) + s\alpha d(Sx_n, Sx_n) + s\alpha H(Sx_n, Ty_n)\]

\[\leq s\alpha d(Sx_n, Sy_{n+1}) + s\alpha d(Sx_n, Sx_n) + s\alpha H(Sx_n, Ty_n)\]

\[\leq s\alpha d(Sx_n, Sy_{n+1}) + s\alpha d(Sx_n, Sx_n) + s\alpha H(Sx_n, Ty_n)\]

Repeating this process \( n - 1 \) times we get

\[d(Sx_{n}, Sy_{n+1}) \leq d(Sx_n, Sx_n) + s\alpha d(Sx_n, Ty_n) + s\varepsilon_n\]

Consequently,

\[d(u, Sx_{n+1}) \leq sd(u, Sx_{n+1}) + sd(Sx_{n+1}, Sx_{n+1})\]

This yields the inequality (14).

Assume that \( \lim_{n \to \infty} Sy_n = u \)

\[\varepsilon_n = H(Sy_{n+1}, Ty_n) \leq sH(Sy_{n+1}, u) + H(u, Ty_n)\]

\[\leq sd(Sy_{n+1}, u) + s\alpha d(u, Sy_n) + s\alpha H(Su, Tu)\]

Letting \( n \to \infty \) we obtain \( \varepsilon_n \to 0 \)

Suppose \( \lim_{n \to \infty} \varepsilon_n = 0 \) since \( 0 \leq \alpha < 1 \) and \( \lim_{n \to \infty} Sx_n = u \)
First two terms on the right hand side (14) vanish in the limit. Applying lemma (2.3) to the last term of inequality (14) we see that

\[ \lim s^2 \sum_{r=0}^{\infty} (s\alpha)^{-r} e_r = 0 \]

Finally we show that,

\[ \lim s^3 \alpha \sum_{r=0}^{n} (s\alpha)^{-r} d( Sx_r, Tx_r ) = 0 \]

Let \( A \) denote the lower triangular matrix with entries \( a_{nr} = (s\alpha)^{-r} \), then

\[ \lim a_{nr} = 0 \quad \text{for each} \quad r \quad \text{and} \]

\[ \lim \sum_{r=0}^{\infty} a_{nr} = \frac{1}{1-s\alpha} \quad \text{therefore,} \quad A \quad \text{is multiplicative} \quad \text{i.e} \quad \text{for any converge sequence} \]

\[ \{ s_n \}, \lim A(s_n) = \frac{1}{1-s\alpha} \lim s_n, \quad \text{thus,} \]

\[ \lim s^3 \alpha \sum_{r=0}^{n} (s\alpha)^{-r} d( Sx_r, Tx_r ) = 0 \]

(\text{cf. Czerwik 1997 p. 692}).

\textbf{Remark 3.1:}

Theorem (3.2) with \( S = I \) (identity map) generalize the main result of Singh et al. 2005.

\textbf{Corollary 3.1:} Singh et al. 2005.

Let \( (X, d) \) be a complete b-metric space and let \( T : X \rightarrow CL( X ) \) satisfy condition (11) with \( S = I \) (identity map), let \( x_o \in X \) and \( \{ x_n \} \) be an orbit for \( T \) at \( x_n \), i.e. \( x_{n+1} \in Tx_n \quad n = 0, 1, 2, \ldots \) and \( \{ x_n \} \) converge to a fixed point \( u \) of \( T \). Let \( \{ y_n \} \) be a sequence in \( X \) and set \( \epsilon_n = H( y_{n+1}, Ty_n ), \quad n = 0, 1, 2, \ldots \),

\[ d(u, Sy_n) \leq s(s\alpha)^{n+1} d(x_o, x_n) + s^2 m^\alpha \sum_{i=0}^{n} (s\alpha)^{i-1} d(x_i, x_{i+1}) + s^2 \sum_{i=0}^{n} (s\alpha)^{-r} \epsilon_n \]

Where \( m = (1-s\alpha)^{-1} \).

If \( Tu \) is singleton, then \( \lim y_n = u \) iff

\[ \lim \epsilon_n = 0 \]

\textbf{Corollary 3.2:} (Czerwik et al. 2002)

Let \( (X, d) \) be a complete b-metric space and let \( T : X \rightarrow CL( X ) \) satisfy

\[ H(Tx, Ty) \leq \alpha d(x, y), \quad x, y \in \lambda \]

where \( 0 < \alpha < s^{-1} \) let \( x_o \in X \) and \( \{ x_n \} \) be an orbit of \( T \) at \( x_o \), i.e. \( x_{n+1} \in Tx_n, \quad n = 0, 1, 2, \ldots \) and \( \{ x_n \} \) converge to fixed point \( u \) of \( T \), moreover let \( \{ y_n \} \) be a sequence in \( X \) and set \( \epsilon_n = H( y_{n+1}, Ty_n ), \quad n = 0, 1, 2, \ldots \),

\[ d(u, y_{n+1}) \leq s d(u, x_{n+1}) + s(s\alpha)^{n+1} d(x_o, y_n) + s^2 \sum_{i=0}^{n} \sum_{j=0}^{n} (s\alpha)^{j-1} \epsilon_n \epsilon_j \]

If moreover, \( Tu \) is singleton, then \( \lim y_n = u \) iff \( \lim \epsilon_n = 0 \)

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Remark 3.2: 1- Theorem (3.2) generalize theorem 4.3 of Singh et al. 2008. when $S$ and $T$ are single valued maps.

2- If $s = I$, $S = Id$ (the identity map), $S$ and $T$ are single valued map in the theorem 3.2, we obtain the main result of Osilike 1995.

Theorem 3.3: let $(X,d)$ be a $b$-metric and $Y \subseteq X$ let $T : Y \to \text{CL}(X)$, $S : Y \to X$ such that $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of $X$. Let $z$ be a coincidence point of $T$ and $S$, i.e. $u = Sz \in Tz$ generated by $Sx_{n+1} \in Tx_n$ converge to $u$. Let $\{Sy_n\} \subseteq X$ and define

$$\varepsilon_n = H(Sy_n,Ty_n), \quad n = 0,1,2,\ldots$$

If $S$ and $T$ satisfies (12) with $s^2q < 1$, then

$$d(u,Sy_n) \leq d(u,Sx_{n+1}) + s(\varepsilon_n)^{2q}d(Sx_n,Sy_n) + s^2L\sum_{i=0}^{n-1}(\varepsilon_i)^{2q}\varepsilon_i + s^2\sum_{i=0}^{n-1}(\varepsilon_i)^{2q}\varepsilon_i$$

$$\lim_{n \to \infty} Sy_n = u \iff \lim_{n \to \infty} \varepsilon_n = 0. \quad (17)$$

Proof: In view of the fact that $Sx_{n+1} \in Tx_n$ for $n = 0,1,2,\ldots$ and the definition of $H$, we get

$$d(Sx_{n+1},Sy_{n+1}) \leq H(Tx_n,Sy_{n+1}).$$

Now, since $H$ is a $b$-metric, so for any $A,B,C \in \text{CL}(X)$ we have

$$H(A,C) \leq s[H(A,B)+H(B,C)]$$

therefore by (12) we obtain

$$d(Sx_{n+1},Sy_{n+1}) \leq s[H(Tx_n,Ty_n)+H(Ty_n,Sy_{n+1})]$$

$$\leq s[2d(Sx_n,Ty_n)+Ld(Sx_{n+1},Tx_n)] + s\varepsilon_n$$

$$\leq (s^2q)^2d(Sx_{n+1},Sy_{n+1}) + s^2qLd(Sx_{n+1},Tx_n) + s^2Ld(Sx_n,Tx_n) + s^2q\varepsilon_{n-1} + s\varepsilon_n$$

Since

Repeating this process $n - 1$ times, we get

$$d(u,Sy_n) \leq d(u,Sx_{n+1}) + s(\varepsilon_n)^{2q}d(Sx_n,Sy_n) + s^2L\sum_{i=0}^{n-1}(\varepsilon_i)^{2q}\varepsilon_i + s^2\sum_{i=0}^{n-1}(\varepsilon_i)^{2q}\varepsilon_i$$

The proof (17) is similar to (15) in Theorem 3.2.

Remark 3.3:

1: In Theorem 3.3 when we put $S$ and $T$ maps on arbitrary set $Y$ with values in $X$ we get Theorem 4.2 in Singh and Prassad 2008 and when $s$ generalize the main result of Singh et al. 2005.

2: Theorem 3.3 generalize the main result of Rhoades 1993 in case of single value map and when $s = I$ and $S = Id$ (Identity map) under the following condition:

$$d(Tx,Ty) \leq q\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$$

We remark that the above condition implies (12). For more detail see Rhoades 1977.
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استقرار العمليات المكررة للدوال الهجينية في الفضاء المتري – ب

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