SIMULTANEOUS APPROXIMATION BY A NEW SEQUENCE OF
SZÃSZ–BETA TYPE OPERATORS

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ABSTRACT. In this paper, we study some direct results in simultaneous approximation for a new sequence of linear positive operators $M_n(f(t);x)$ of Szãsz-Beta type operators. First, we establish the basic pointwise convergence theorem and then proceed to discuss the Voronovaskaja-type asymptotic formula. Finally, we obtain an error estimate in terms of modulus of continuity of the function being approximated.
1. INTRODUCTION

In [3] Gupta and others studied some direct results in simultaneous approximation for the sequence:

\[ B_n(f(t);x) = \sum_{k=0}^{\infty} q_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t) \, dt, \]

where \( x, t \in [0, \infty) \), \( q_{n,k}(x) = \frac{e^{-nx}(nx)^k}{k!} \) and \( b_{n,k}(t) = \frac{\Gamma(n+k+1)}{\Gamma(n)\Gamma(k+1)} t^k (1+t)^{-(n+k+1)}. \)

After that, Agrawal and Thamer [1] applied the technique of linear combination introduced by May [4] and Rathore [5] for the sequence \( B_n(f(t);x). \)

Recently, Gupta and Lupas [2] studied some direct results for a sequence of mixed Beta-Szász type operator defined as

\[ L_n(f(t);x) = \sum_{k=1}^{\infty} b_{n,k}(x) \int_0^{\infty} q_{n,k-1}(t) f(t) \, dt + (1+x)^{-n-1} f(0). \]

In this paper, we introduce a new sequence of linear positive operators \( M_n(f(t);x) \) of Szász-Beta type operators to approximate a function \( f(x) \) belongs to the space \( C_\alpha [0, \infty) = \{ f \in C[0, \infty) : \| f \|_{C_\alpha} \leq C(1+t)^\alpha \text{ for some } C > 0, \alpha > 0 \} \), as follows:

\[ M_n(f(t);x) = \sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t) f(t) \, dt + e^{-nx} f(0), \]

We may also write the operator (1) as \( M_n(f(t);x) = \int_0^{\infty} W_n(t,x)f(t) \, dt \) where

\[ W_n(t,x) = \sum_{k=1}^{\infty} q_{n,k}(x) b_{n,k-1}(t) + e^{-nx} \delta(t), \quad \delta(t) \text{ being the Dirac-delta function.} \]

The space \( C_\alpha [0, \infty) \) is normed by \( \| f \|_{C_\alpha} = \sup_{0 \leq t < \infty} | f(t) | (1+t)^{-\alpha}. \)

Throughout this paper, we assume that \( C \) denotes a positive constant not necessarily the same at all occurrence and \([\beta]\) denotes the integer part of \( \beta \).

2. PRELIMINARY RESULTS

For \( f \in C[0, \infty) \) the Szász operators is defined as

\[ S_n(f;x) = \sum_{k=1}^{\infty} q_{n,k}(x) f\left( \frac{k}{n} \right), \quad x \in [0, \infty) \] and for \( m \in N^0 \) (the set of nonnegative integers), the \( m \)-th order moment of the Szász operators is defined as

\[ \mu_{n,m}(x) = \sum_{k=0}^{\infty} q_{n,k}(x) \left( \frac{k}{n} - x \right)^m. \]

**Lemma 2.1.** [3] For \( m \in N^0 \), the function \( \mu_{n,m}(x) \) defined above, has the following properties:
(i) $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = 0$, and the recurrence relation is

$$n\mu_{n,m+1}(x) = x\left(\mu_{n,m}'(x) + m\mu_{n,m-1}(x)\right), \quad m \geq 1;$$

(ii) $\mu_{n,m}(x)$ is a polynomial in $x$ of degree at most $\lfloor m/2 \rfloor$;

(iii) For every $x \in [0, \infty)$, $\mu_{n,m}(x) = O\left(n^{-\lfloor (m+1)/2 \rfloor}\right)$.

From above lemma, we get

$$\sum_{k=1}^{\infty} q_{n,k}(x)(k-nx)^{2j} = n^{2j}\left(\mu_{n,2j}(x) - (-x)^{2j}e^{-nx}\right)$$

$$= n^{2j}\left\{O(n^{-j}) + O(n^{-s})\right\} \quad \text{(for any } s > 0)$$

$$= O(n^{-j}) \quad \text{(if } s \geq j \text{).}$$

For $m \in \mathbb{N}^0$, the $m$-th order moment $T_{n,m}(x)$ for the operators (1) is defined as:

$$T_{n,m}(x) = M_n\left((t-x)^m; x\right) = \sum_{k=1}^{\infty} q_{n,k}(x)\int_{0}^{\infty} b_{n,k-1}(t)(t-x)^m \, dt + (-x)^m e^{-nx}.$$  

**Lemma 2.2.** For the function $T_{n,m}(x)$, we have $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{x}{n-1}$, $T_{n,2}(x) = \frac{nx^2 + 2nx + 2x^2}{(n-1)(n-2)}$ and there holds the recurrence relation:

$$(n-m-1)T_{n,m+1}(x) = xT_{n,m}'(x) + \left((2x+1)m+x\right)T_{n,m}(x) + mx(x+2)T_{n,m-1}(x), \quad n > m + 1.$$  

Further, we have the following consequences of $T_{n,m}(x)$:

(i) $T_{n,m}(x)$ is a polynomial in $x$ of degree exactly $m$;

(ii) For every $x \in [0, \infty)$, $T_{n,m}(x) = O\left(n^{-\lfloor (m+1)/2 \rfloor}\right)$.

**Proof:** By direct computation, we have $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{x}{n-1}$, and $T_{n,2}(x) = \frac{nx^2 + 2nx + 2x^2}{(n-1)(n-2)}$. Next, we prove (3). For $x = 0$ it clearly holds. For $x \in (0, \infty)$, we have

$$T_{n,m}'(x) = \sum_{k=1}^{\infty} q_{n,k}'(x)\int_{0}^{\infty} b_{n,k-1}(t)(t-x)^m \, dt - n(-x)^m e^{-nx} - mT_{n,m-1}(x).$$

Using the relations $xq_{n,k}'(x) = (k-nx)q_{n,k}(x)$ and $t(1+t)b_{n,k}'(t) = (k-(n+1)t)b_{n,k}(t)$, we get:

$$xT_{n,m}'(x) = \sum_{k=1}^{\infty} (k-nx)q_{n,k}(x)\int_{0}^{\infty} b_{n,k-1}(t)(t-x)^m \, dt + n(-x)^{m+1} e^{-nx} - mxT_{n,m-1}(x)$$
\[= \sum_{k=1}^{\infty} q_{n,k}(x) \int_{0}^{\infty} t(1+t)b''_{n,k-1}(t)(t-x)^m dt + (n+1)T_{n,m+1}(x) - (-x)^{m+1}e^{-nx} + (x+1)T_{n,m}(x) - (x+1)(-x)^m e^{-nx} - mxT_{n,m-1}(x).\]

By using the identity \( t(1+t) = (t-x)^2 + (1+2x)(t-x) + x(1+x) \), we have

\[xT'_{n,m}(x) = \sum_{k=1}^{\infty} q_{n,k}(x) \int_{0}^{\infty} b''_{n,k-1}(t)(t-x)^{m+2} dt + (1+2x)\sum_{k=1}^{\infty} q_{n,k}(x) \int_{0}^{\infty} b'_{n,k-1}(t)(t-x)^{m+1} dt + x(1+x)\sum_{k=1}^{\infty} q_{n,k}(x) \int_{0}^{\infty} b'_{n,k-1}(t)(t-x)^m dt + (n+1)T_{n,m+1}(x) + (1+x)T_{n,m}(x) - mxT_{n,m-1}(x) - (-x)^m e^{-nx}.\]

Integrating by parts, we get

\[xT'_{n,n}(x) = (n-m-1)T_{n,m+1}(x) - (m+x+2mx)T_{n,m}(x) - mx(x+2)T_{n,m-1}(x)\]

from which (3) is immediate.

From the values of \( T_{n,0}(x) \) and \( T_{n,1}(x) \), it is clear that the consequences (i) and (ii) hold for \( m = 0 \) and \( m = 1 \). By using (3) and the induction on \( m \) the proof of consequences (i) and (ii) follows, hence the details are omitted.

\[\blacksquare\]

From the above lemma, we have

\[\sum_{k=1}^{\infty} q_{n,k}(x) \int_{0}^{\infty} b''_{n,k-1}(t)(t-x)^{2r} dt = T_{n,2r} - (-x)^{2r} e^{-nx} = O(n^{-r}) + O(n^{-s}) \text{ (for any } s > 0 \text{)} \]
\[= O(n^{-r}) \text{ (if } s \geq r \text{).}\]

**Lemma 2.3.** Let \( \delta \) and \( \gamma \) be any two positive real numbers and \([a,b] \subset (0,\infty)\). Then, for any \( s > 0 \), we have

\[\left\| \int_{|t-x|\geq\delta} W_n(t,x) t^s \, dt \right\|_{C[a,b]} = O(n^{-s}).\]

Making use of Schwarz inequality for integration and then for summation and (4), the proof of the lemma easily follows.

**Lemma 2.4.** [3] There exist polynomials \( Q_{i,j,r}(x) \) independent of \( n \) and \( k \) such that
\[ x^r D^r (q_{n,k}(x)) = \sum_{2i+j \leq r} n^i (k-nx)^j Q_{i,j,r}(x) q_{n,k}(x), \text{ where } D = \frac{d}{dx}. \]

3. MAIN RESULTS

Firstly, we show that the derivative \( M_n^{(r)}(f(t);x) \) is an approximation process for \( f^{(r)}(x), r = 1, 2, \ldots. \)

**Theorem 3.1.** If \( r \in \mathbb{N}, f \in C_\alpha [0, \infty) \) for some \( \alpha > 0 \) and \( f^{(r)} \) exists at a point \( x \in (0, \infty), \) then

\[ \lim_{n \to \infty} M_n^{(r)}(f(t);x) = f^{(r)}(x). \]

Further, if \( f^{(r)} \) exists and is continuous on \( (a-\eta, b+\eta) \subset (0, \infty), \) \( \eta > 0, \) then (5) holds uniformly in \([a, b].\)

**Proof:** By Taylor's expansion of \( f, \) we have

\[ f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} (t-x)^i + \varepsilon(t,x)(t-x)^r, \]

where, \( \varepsilon(t,x) \to 0 \) as \( t \to x \). Hence

\[ M_n^{(r)}(f(t);x) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_{0}^{\infty} W_n^{(r)}(t,x)(t-x)^i \, dt + \int_{0}^{\infty} W_n^{(r)}(t,x) \varepsilon(t,x)(t-x)^r \, dt \]

\[ \equiv I_1 + I_2. \]

Now, using Lemma 2.2 we get that \( M_n(t^m; x) \) is a polynomial in \( x \) of degree exactly \( m, \) for all \( m \in \mathbb{N}^0. \) Further, we can write it as:

\[ M_n(t^m; x) = \frac{(n-m-1)! n^m}{(n-1)!} x^m + \frac{(n-m-1)! n^{m-1}}{(n-1)!} m(m-1) x^{m-1} + O(n^{-2}). \]

Therefore,

\[ I_1 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \int_{0}^{\infty} W_n^{(r)}(t,x) t^j \, dt \]

\[ = \frac{f^{(r)}(x)}{r!} \binom{n-r-1}{(n-1)!} n^r \rightarrow f^{(r)}(x) \text{ as } n \to \infty. \]

Next, making use of Lemma 2.4 we have

\[ |I_2| \leq \sum_{2i+j \leq r} n^i \sum_{k=1}^{\infty} q_{n,k}(x) \int_{0}^{\infty} b_{n,k-1}(t) |\varepsilon(t,x)| |t-x|^r \, dt + (nx)^r e^{-nx} \varepsilon(0,x) \]

\[ \equiv I_3 + I_4. \]
Since \( \varepsilon(t, x) \to 0 \) as \( t \to x \), then for a given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( |\varepsilon(t, x)| < \varepsilon \), whenever \( 0 < |t - x| < \delta \). For \( |t - x| \geq \delta \), there exists a constant \( C > 0 \) such that \( |\varepsilon(t, x)(t - x)^r| \leq C|t - x|^r \).

Now, since \( |\varepsilon(t, x)| < \varepsilon \), whenever \( \delta < \gamma \), there exists a \( \delta > 0 \) such that \( |\varepsilon(t, x)| < \varepsilon \), whenever \( \delta < \gamma \).

Now, applying Schwartz inequality for integration and then for summation, (2) and (4) we led to

\[
I_5 \leq \varepsilon C \sum_{2i + jsr, i, j \geq 0} n^i \sum_{k=1}^{\infty} q_{n,k}(x) \left( \int_0^\infty b_{n,k-1}(t)(t - x)^{2r} \, dt \right)^{1/2} \left( \int_0^\infty b_{n,k-1}(t) \, dt \right)^{1/2} \]

(since \( \int_0^\infty b_{n,k-1}(t) \, dt = 1 \))

\[
\leq \varepsilon C \sum_{2i + jsr, i, j \geq 0} n^i O(n^{-r/2}) \sum_{k=1}^{\infty} q_{n,k}(x) \left( \int_0^\infty b_{n,k-1}(t)(t - x)^{2r} \, dt \right)^{1/2} \left( \int_0^\infty b_{n,k-1}(t) \, dt \right)^{1/2} \]

\[
\leq \varepsilon C O(n^{-r/2}) \sum_{2i + jsr, i, j \geq 0} n^i = O(1). \]

Again using Schwartz inequality for integration and then for summation, in view of (2) and Lemma 2.3, we have

\[
I_6 \leq C \sum_{2i + jsr, i, j \geq 0} n^i \sum_{k=1}^{\infty} q_{n,k}(x) \left( \int_0^\infty b_{n,k-1}(t)(t - x)^{2r} \, dt \right)^{1/2} \left( \int_0^\infty b_{n,k-1}(t) \, dt \right)^{1/2} \]

\[
\leq C \sum_{2i + jsr, i, j \geq 0} n^i \sum_{k=1}^{\infty} q_{n,k}(x) \left( \int_0^\infty b_{n,k-1}(t)(t - x)^{2r} \, dt \right)^{1/2} \left( \int_0^\infty b_{n,k-1}(t) \, dt \right)^{1/2} \]

\[
\leq C \sum_{2i + jsr, i, j \geq 0} n^i \left( \sum_{k=1}^{\infty} q_{n,k}(x) \right) \left( \int_0^\infty b_{n,k-1}(t)(t - x)^{2r} \, dt \right)^{1/2} \left( \int_0^\infty b_{n,k-1}(t) \, dt \right)^{1/2} \]

\[
\leq O(n^{-r}) \sum_{2i + jsr, i, j \geq 0} n^i \ (\text{for any } s > 0) \]
Now, since $\varepsilon > 0$ is arbitrary, it follows that $I_3 = o(1)$. Also, $I_4 \to 0$ as $n \to \infty$ and hence $I_2 = o(1)$, combining the estimates of $I_1$ and $I_2$, we obtain (5).

To prove the uniformity assertion, it sufficient to remark that $\delta(\varepsilon)$ in above proof can be chosen to be independent of $x \in [a,b]$ and also that the other estimates holds uniformly in $[a,b]$.

Our next theorem is a Voronovaskaja-type asymptotic formula for the operators $M_n^{(r)}(f(t); x)$, $r = 1, 2, \ldots$.

**THEOREM 3.2.** Let $f \in C_{\alpha}[0, \infty)$ for some $\alpha > 0$. If $f^{(r+2)}$ exists at a point $x \in (0, \infty)$, then

$$
\lim_{n \to 0} n \left( M_n^{(r)}(f(t); x) - f^{(r)}(x) \right) = \frac{r(r+1)}{2} f^{(r)}(x) + ((r+1)x + r) f^{(r+1)}(x) + \frac{1}{2} x(x+2) f^{(r+2)}(x).
$$

Further, if $f^{(r+2)}$ exists and is continuous on the interval $(a-\eta, b+\eta) \subset (0, \infty)$, $\eta > 0$, then (7) holds uniformly on $[a,b]$.

**Proof:** By the Taylor’s expansion of $f(t)$, we get

$$
M_n^{(r)}(f(t); x) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} M_n^{(r)}(t-x)^i; x) + M_n^{(r)}(\varepsilon(t,x)(t-x)^{r+2}; x)
$$

$$
:= I_1 + I_2,
$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.

By Lemma 2.2 and (6), we have

$$
I_1 = \sum_{i=r}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=r}^{i} \binom{i}{j} (-x)^{i-j} M_n^{(r)}(t^j; x)
$$

$$
= \frac{f^{(r)}(x)}{r!} M_n^{(r)}(t^r; x) + \frac{f^{(r+1)}(x)}{(r+1)!} \left[ (r+1)x M_n^{(r)}(t^r; x) + M_n^{(r)}(t^{r+1}; x) \right]
$$

$$
+ \frac{f^{(r+2)}(x)}{(r+2)!} \left[ \frac{(r+2)(r+1)}{2} x^2 M_n^{(r)}(t^r; x) + (r+2)(-x) M_n^{(r)}(t^{r+1}; x) + M_n^{(r)}(t^{r+2}; x) \right]
$$

$$
= f^{(r)}(x) \left( \frac{(n-r-1)! n^r}{(n-1)!} \right)
$$

$$
+ \frac{f^{(r+1)}(x)}{(r+1)} \left[ (r+1)(-x) \frac{(n-r-1)! n^r}{(n-1)!} \right]
$$
+ \left( \frac{(n-r-2)!n^{r+1}}{(n-1)!} (r+1)x + \frac{(n-r-2)!n^{r}}{(n-1)!} (r+1)r! \right)
\right) \\
+ \frac{f^{(r+2)}(x)}{(r+2)!} \left\{ \frac{(r+1)(r+2)}{2} x^2 \left( \frac{(n-r-1)!n^{r}}{(n-1)!} - r! \right)
\right) \\
+ (r+2)(-x) \left\{ \frac{(n-r-2)!n^{r+1}}{(n-1)!} (r+1)x + \frac{(n-r-2)!n^{r}}{(n-1)!} (r+1)r! \right) \\
\right) \\
+ \left( \frac{(n-r-3)!n^{r+2}}{(n-1)!} \cdot \frac{(r+2)}{2} x^2 + \frac{(n-r-3)!n^{r+1}}{(n-1)!} (r+2)(r+1)(r+1)x \right) + O(n^{-2}).

Hence in order to prove (7) it suffices to show that \( nI_2 \to 0 \) as \( n \to \infty \), which follows on proceeding along the lines of proof of \( I_2 \to 0 \) as \( n \to \infty \) in Theorem 3.1. The uniformity assertion follows as in the proof of Theorem 3.1.

Finally, we present a theorem which gives as an estimate of the degree of approximation by \( M_n^{(r)}(.;x) \) for smooth functions.

**Theorem 3.3.** Let \( f \in C_{\alpha}[0,\infty) \) for some \( \alpha > 0 \) and \( r \leq q \leq r + 2 \). If \( f^{(q)} \) exists and is continuous on \( (a-\eta,b+\eta) \subset (0,\infty), \eta > 0 \), then for sufficiently large \( n \),

\[
\left\| M_n^{(r)}(f(t);x) - f^{(r)}(x) \right\|_{C[a,b]} \leq C_1 n^{-1} \sum_{i=r}^q \left\| f^{(i)} \right\|_{C[a,b]} + C_2 n^{-1/2} \omega_f(\delta) n^{-1/2} + O(n^{-2})
\]

where \( C_1, C_2 \) are constants independent of \( f \) and \( n \), \( \omega_f(\delta) \) is the modulus of continuity of \( f \) on \( (a-\eta,b+\eta) \), and \( \left\| \cdot \right\|_{C[a,b]} \) denotes the sup-norm on \( [a,b] \).

**Proof.** By Taylor's expansion of \( f \), we have

\[
f(t) = \sum_{i=0}^q \frac{f^{(i)}(x)}{i!} (t-x)^i + \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!} (t-x)^q \chi(t) + h(t,x)(1 - \chi(t)),
\]

where \( \xi \) lies between \( t,x \), and \( \chi(t) \) is the characteristic function of the interval \( (a-\eta,b+\eta) \). Now,

\[
M_n^{(r)}(f(t);x) - f^{(r)}(x) = \left\{ \sum_{i=0}^q \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x) \right\} \\
+ \int_0^\infty W_n^{(r)}(t,x) \left\{ \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!} (t-x)^q \chi(t) \right\} dt + \int_0^\infty W_n^{(r)}(t,x) h(t,x)(1 - \chi(t)) dt
\]

\[= I_1 + I_2 + I_3.\]
By using Lemma 2.2 and (6), we get
\[
I_1 = \sum_{i=r}^{q} \frac{f^{(i)}(x)}{i!} \sum_{j=r}^{k} \left(-x\right)^{i-j} \frac{d^j}{dx^j} \left(\frac{(n-j-1)! \ n^j}{(n-1)!} x^j + \frac{(n-j-1)! \ n^{j-1}}{(n-1)!} j(j-1)x^{j-1} + O(n^{-2})\right) - f^{(r)}(x).
\]

Consequently,
\[
\|I_1\|_{C[a,b]} \leq C_1 n^{-1} \sum_{i=r}^{q} \|f^{(i)}\|_{C[a,b]} + O(n^{-2}), \text{ uniformly on } [a,b].
\]

To estimate \(I_2\), we proceed as follows:
\[
|I_2| \leq \left\|W_n^{(r)}(t, x)\right\| \left\{\left\|f^{(q)}(\xi) - f^{(q)}(x)\right\| \xi - x^q \chi(t)\right\} dt
\]
\[
\leq \frac{\omega^{(q)}(\delta)}{q!} \int_0^\infty \left|W_n^{(r)}(t, x)\right| (1 + \frac{|t-x|}{\delta}) |t-x|^q dt
\]
\[
\leq \frac{\omega^{(q)}(\delta)}{q!} \left[\sum_{k=1}^{\infty} q_{n,k}^{(r)}(x) \int_0^\infty b_{n,k-1}(t) |t-x|^q + \delta^{-1} |t-x|^{q+1}\right] dt
\]
\[
+ (-n)^r e^{-nx} (x^q + \delta^{-1}x^{q+1}), \quad \delta > 0.
\]

Now, for \(s = 0, 1, 2, \ldots\), using Schwartz inequality for integration and then for summation, (2) and (4), we have
\[
\sum_{k=1}^{\infty} q_{n,k}(x)|k-nx| \left|\int_0^\infty b_{n,k-1}(t) |t-x|^s dt\right| \leq \sum_{k=1}^{\infty} q_{n,k}(x)|k-nx| \left\{ \int_0^\infty b_{n,k-1}(t) dt \right\}^{1/2}
\]
\[
\times \left\{ \int_0^\infty q_{n,k}(t)(t-x)^{2s} dt \right\}^{1/2}
\]
\[
\leq \left( \sum_{k=1}^{\infty} q_{n,k}(x)(k-nx)^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} q_{n,k}(x) \int_0^\infty b_{n,k-1}(t)(t-x)^{2s} dt \right)^{1/2}
\]
\[
= O(n^{l/2})O(n^{-s/2})
\]
\[
= O(n^{(l-s)/2}), \text{ uniformly on } [a,b].
\]

Therefore, by Lemma 2.4 and (8), we get
\[
\sum_{k=1}^{\infty} q_{n,k}^{(r)}(x) \left|\int_0^\infty b_{n,k-1}(t) |t-x|^s dt\right| \leq \sum_{k=1}^{\infty} n^i |k-nx| \left|\frac{\partial_i r(x)}{x^r}\right| q_{n,k}(x)
\]
\[
\times \left( \int_0^\infty b_{n,k-1}(t) |t-x|^s dt \right)
\]

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\[
\begin{align*}
\leq & \left( \sup_{2i+j \leq r \atop i,j \geq 0} \sup_{x \in [a,b]} \frac{|Q_{i,j,r}(x)|}{x^r} \right) \sum_{2i+j \leq r \atop i,j \geq 0} n^i \left( \sum_{k=1}^{\infty} q_{n,k}(x) |k - nx|^j \int_0^{\infty} b_{n,k-1}(t) |t - x|^s \, dt \right) \\
= & C \sum_{2i+j \leq r \atop i,j \geq 0} n^i O(n^{(j-s)/2}) = O(n^{(r-s)/2}), \text{uniformly on } [a,b].
\end{align*}
\]

(since \( \sup_{2i+j \leq r \atop i,j \geq 0} \sup_{x \in [a,b]} \frac{|Q_{i,j,r}(x)|}{x^r} = M(x) \) but fixed)

Choosing \( \delta = n^{-1/2} \) and applying (9), we are led to

\[
\begin{align*}
\|f\|_{C[a,b]} & \leq \frac{\omega_{(q)}(n^{-1/2})}{q!} \left[ O(n^{(r-q)/2}) + n^{1/2} O(n^{(r-q-1)/2}) + O(n^{-m}) \right], \text{ (for any } m > 0) \\
& \leq C_2 n^{-(r-q)/2} \omega_{(q)}(n^{-1/2}).
\end{align*}
\]

Since \( t \in [0,\infty) \setminus (a-\eta, b+\eta) \), we can choose \( \delta > 0 \) in such a way that \( |t - x| \geq \delta \) for all \( x \in [a,b] \). Thus, by Lemmas 2.3 and 2.4, we obtain

\[
|I_3| \leq \sum_{k=1}^{\infty} \sum_{2i+j \leq r \atop i,j \geq 0} n^i |k - nx|^j \frac{|Q_{i,j,r}(x)|}{x^r} q_{n,k}(x) \int b_{n,k-1}(t) |h(t,x)| |t - x|^s \, dt + (-n)^r e^{-nx} |h(0,x)|.
\]

For \( |t - x| \geq \delta \), we can find a constant \( C \) such that \( |h(t,x)| \leq C |t - x|^\alpha \). Hence, using Schwarz inequality for integration and then for summation , (2), (4), it easily follows that \( I_3 = O(n^{-s}) \) for any \( s > 0 \), uniformly on \([a,b]\).

Combining the estimates of \( I_1, I_2, I_3 \), the required result is immediate.

\[\blacksquare\]

**REFERENCES**


